Freecalc
Homework on Lecture 12
Quiz date will be announced in class

1. Find whether the series is convergent or divergent using an appropriate test.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \ln n.$

   (b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}.$

   Solution. 1.a. $\displaystyle\lim_{n\to\infty}(-1)^n \ln n$ does not exist and therefore the sum is not convergent.

   Solution. 1.b. For $n > 2$, we have that $\ln n$ is a positive increasing function and therefore $\frac{1}{\ln n}$ is a decreasing positive function. Furthermore $\displaystyle\lim_{n\to\infty}\frac{1}{\ln n} = 0$. Therefore the series is convergent by the alternating series test.

2. (The last problem will NOT appear on the quiz) Use integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n+1}.$

   answer: divergent

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2+n^3}.$

   answer: convergent

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3}{3n^5+n}$

   answer: convergent, can use limit comparison test

   (d) $\displaystyle\sum_{2}^{\infty}\frac{1}{n\ln n}$

   (e) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(2n+1)\ln(n)}.$

   answer: divergent

   (f) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

   answer: convergent, can use integral test

   (g) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(2n+1)(\ln(n))^2}.$

   answer: convergent

   (h) Determine all values of p, q r for which the series
   $$\sum_{n=30}^{\infty}\frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$
   is convergent.

   Solution. 2d.

$$\begin{aligned}\int_2^{\infty}\frac{1}{x\ln x}\mathrm{d}x &= \lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}\mathrm{d}x\\ &= \lim_{t\to\infty}\int_2^t\frac{1}{\ln x}\mathrm{d}(\ln x)\\ &= \lim_{t\to\infty}\int_2^t \mathrm{d}(\ln(\ln x))\\ &= \lim_{t\to\infty}\left[\ln(\ln x)\right]_{x=2}^{x=t}\\ &= \lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)\\ &= \infty,\end{aligned}$$

therefore the integral is divergent (and diverges to $+\infty$).
The function $\frac{1}{x\ln x}$ is decreasing, as for $x > 2$, it is the quotient of 1 by increasing positive functions. $\frac{1}{x\ln x}$ tends to 0 as $x\to\infty$, and therefore the integral criterion implies that $\sum_{2}^{\infty}\frac{1}{n\ln n}$ is divergent.

Solution. 2e

The integral criterion appears to be of little help: the improper integral $\int \frac{1}{(2x+1)\ln x} \mathrm{d}x$ cannot be integrated algebraically with any of the techniques we have studied so far. Therefore it makes sense to try to solve this problem using a comparison test.

The "leading term"[1] of the denominator of $\frac{1}{(2n+1)\ln n} = \frac{1}{2n\ln n + \ln n}$ is $2n\ln n$. Therefore it makes sense to compare - or limit-compare - with $\frac{1}{n\ln n}$.

We will use the Limit Comparison Test for the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \frac{1}{(2n+1)\ln n}$ and $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty} \frac{1}{n\ln n}$. Both a_n and b_n are positive (for $n > 2$) and therefore the Limit Comparison Test applies.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(2n+1)\ln n}}{\frac{1}{n\ln n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} \neq 0$, the Limit Comparison Test implies that the series $\sum_{n=2}^{\infty} a_n$ has same convergence/divergence properties as the series $\sum_{n=2}^{\infty} b_n$. In Problem 2d we demonstrated that the series $\sum\limits_{n=2}^{\infty} b_n$ is divergent; therefore the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \frac{1}{(2n+1)\ln n}$ is divergent as well.

[1] here we use the expression "leading term" informally - that terminology requires that we speak of rational functions