Freecalc
Homework on Lecture 14
Quiz date will be announced in class

1. Determine the interval of convergence for the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

   answer: $x \in [1,3)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

   answer: $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

   answer: $x \in (2,4]$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   answer: converges for all x

   (e) $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n$

   answer: converges for $|x| < 1$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

   answer: converges for $|x| \in [-1,1)$.

   (g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

   . answer: converges for $|x| \in (-1,1]$

   (h) $\displaystyle\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\displaystyle\binom{q}{n}$ stands for $\displaystyle\frac{q(q-1)\dots(q-n+1)}{n!}$.

   answer: converges for $x \in (-1,1]$.

   Solution. 1.a. We apply the Ratio Test to get that $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2|$. Therefore the power series converges at least on the interval $(1,3)$. When $x=3$, the series becomes $\sum\limits_{n=1}^{\infty}\frac{1}{3\sqrt{n+1}}$, which diverges - this can be seen, for example, by comparing to the p-series $\frac{1}{\sqrt{n}}$. When $x=1$, the series becomes $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{3\sqrt{n+1}}$, which converges by the Alternating Series Test. Our final answer $x \in [1,3)$.

2. Determine the interval of convergence for the following power series. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$

   answer: $x \in [0.9, 1.1]$

   (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$

   answer: $x \in (-2,0]$

3. (a) Find the Maclaurin series for xe^{x^3}.

   answer: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!}$

(b) Use your series to find the Maclaurin series of $\int xe^{x^3}\,dx$.

answer: $C + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)n!}$ note the integral can't be integrated with elementary functions.

4. Find the Maclaurin series of the function. The answer key has not been proofread, use with caution.

(a) $\frac{1}{2x+3}$.

answer: $\frac{1}{3}\left(1 - \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 - \left(\frac{2x}{3}\right)^3 + \dots\right) = \sum_{0}^{\infty} \frac{(-1)^n}{3}\left(\frac{2}{3}\right)^n x^n$

(b) $\frac{1}{(1-x)^2}$.

answer: $1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty}(n+1)x^n$

(c) $\frac{1}{(1-x)^3}$.

answer: $\frac{1}{2}\left(2 + 6x + 12x^2 + \dots + n(n-1)x^{n-2} + \dots\right) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

(d) xe^{-2x}.

answer: $\sum_{n=0}^{\infty}(-1)^n 2^n x^{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}2^{n-1}x^n$

5. Compute the Maclaurin series of the function. Please post on piazza if you discover errors in the answer key.

(a) e^x.

answer: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) e^{2x}.

answer: $e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

(c) e^{x^2}.

answer: $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

(d) e^{-3x^2}.

answer: $e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{2n}}{n!}$

(e) x^2e^{2x}.

answer: $e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+2}}{n!}$

(f) $\sin x$.

answer: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(g) $\cos x$.

answer: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(h) $\sin(2x)$.

answer: $\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!}$

(i) $\cos(2x)$.

answer: $\cos(2x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{x^{2n}}{(2n)!}$

(j) $\cos^2(x)$.

answer: $\cos^2 x = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n 2^{2n-1} \frac{x^{2n}}{(2n)!}$

(k) $x \sin x$.

answer: $x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$

6. Compute the Maclaurin series of the function. Please post on piazza if you see errors in the answer key.

(a) $\frac{1}{3-x}$.

answer: $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$

(b) $\frac{1}{3-2x}$.

answer: $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$

(c) $\frac{1}{1+x^2}$.

answer: $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

(d) $\frac{1}{1-2x^2}$.

answer: $\sum_{n=0}^{\infty} 2^n x^{2n}$

(e) $\frac{1}{x^2-1}$.

answer: $-\sum_{n=0}^{\infty} x^{2n}$

(f) $\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$.

answer: same as 6.e

(g) $\frac{1}{(1-x)^2}$.

answer: $\sum_{n=0}^{\infty} (n+1)x^n$

(h) $\frac{1}{(1-x)^3}$.

answer: $\frac{1}{2}\sum_{n=0}^{\infty} (n+1)(n+2)x^n = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$

(i) $\ln(1+x)$.

answer: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

(j) $\ln(1-x)$.

answer: $-\sum_{n=1}^{\infty} \frac{x^n}{n}$

(k) $\ln(1-3x)$.

answer: $-\sum_{n=0}^{\infty} \frac{3^n x^n}{n}$

(l) $\ln(1-3x^2)$.

answer: $-\sum_{n=1}^{\infty} \frac{3^n x^{2n}}{n}$

(m) $\ln(3-2x^2)$.

answer: $\ln 3 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n}}{n}$

(n) $x\ln(3-2x^2)$.

answer: $x \ln 3 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n+1}}{n}$

(o) $\arctan x$.

answer: $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

(p) $\arctan(2x)$.

answer: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$

(q) $\arctan\left(2x^2\right)$.

answer: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+2}}{2n+1}$

Solution. 6.m. We solve this problem by using algebraic manipulations and substitutions to reduce it to the already studied power series expansion of $\ln(1-y) = -\sum\limits_{n=1}^{\infty} \frac{y^n}{n}$.

$$
\begin{aligned}
\ln\left(3-2x^2\right) &= \ln\left(3\left(1-\frac{2}{3}x^2\right)\right) \\
&= \ln 3 + \ln\left(1-\frac{2}{3}x^2\right) && \Big| \text{ Set } y = \tfrac{2}{3}x^2 \\
&= \ln 3 + \ln(1-y) \\
&= \ln 3 - \sum_{n=1}^{\infty} \frac{y^n}{n} && \Big| \text{ Substitute back } y = \tfrac{2}{3}x^2 \\
&= \ln 3 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n}}{n} \quad .
\end{aligned}
$$

7. Compute the Maclaurin series of

$$
\left(\frac{1}{(1-x)^k}\right) \quad ,
$$

where $n \geq 1$ is an integer.

Solution. 7 We have that

$$
\begin{array}{lllll}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) & = \frac{(1-x)'}{(1-x)^2} & & = & \frac{1}{(1-x)^2} \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{1-x}\right) & = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^2}\right) & = -2\frac{(1-x)'}{(1-x)^3} & = & \frac{2}{(1-x)^3} \\
\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{1}{1-x}\right) & = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{(1-x)^3}\right) & = 2(-3)\frac{(1-x)'}{(1-x)^4} & = & \frac{2\cdot 3}{(1-x)^4} \\
 & \vdots & & & \\
\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}\left(\frac{1}{1-x}\right) & & & = & \frac{(k-2)!}{(1-x)^{k-1}} \\
\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}\left(\frac{1}{1-x}\right) & = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(k-2)!}{(1-x)^{k-1}}\right) & & = & \frac{(k-1)!}{(1-x)^k} \\
 & \vdots & & &
\end{array}
$$

We can now compute Maclaurin series as follows:

$$
\begin{aligned}
\mathrm{Mc}\left(\frac{1}{(1-x)^k}\right) &= \mathrm{Mc}\left(\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\frac{1}{(1-x)}\right)\right) \\
&= \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\mathrm{Mc}\left(\frac{1}{1-x}\right)\right) \\
&= \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\sum_{n=0}^{\infty} x^n\right) \\
&= \frac{1}{(k-1)!}\left(\sum_{n=0}^{\infty} n(n-1)\dots(n-k+2)x^{n-k+1}\right) && \text{Recall } \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \\
&= \sum_{n=0}^{\infty}\binom{n}{k-1}x^{n-k+1} && \text{Set } n-k+1=m \\
&= \sum_{m=-k+1}^{\infty}\binom{m+k-1}{k-1}x^m && \text{first } k-2 \text{ summands are zero} \\
&= \sum_{m=0}^{\infty}\binom{m+k-1}{k-1}x^m
\end{aligned}
$$

8. Compute the Maclaurin series of

$$(1+x)^q \quad ,$$

where $q \in \mathbb{R}$ is an arbitrary real number.

Solution. 8 Since q does not have to be an integer, we cannot directly relate its power series to the power series of $\frac{1}{1+x}$ or its derivatives. We therefore compute the Maclaurin series directly using their definition.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^q\right) &= q(1+x)^{q-1} \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left((1+x)^q\right) &= q(q-1)(1+x)^{q-2} \\
&\vdots \\
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right) &= q(q-1)(q-2)\dots(q-n+1)(1+x)^{q-n} \quad .
\end{aligned}
$$

Therefore $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right)_{|x=0} = q(q-1)(q-2)\dots(q-n+1)(1+0)^{q-n} = q(q-1)(q-2)\dots(q-n+1)$. Therefore

$$
\begin{aligned}
\mathrm{Mc}\left((1+x)^q\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right)_{|x=0} x^n \\
&= \sum_{n=0}^{\infty}\frac{q(q-1)(q-2)\dots(q-n+1)}{n!}x^n = \sum_{n=0}^{\infty}\binom{q}{n}x^n \quad .
\end{aligned}
\tag{1}
$$

For the last equality we recall the definition of binomial coefficient $\binom{q}{n} = \frac{q(q-1)\dots(q-n+1)}{n!}$ and that it allows for q to be an arbitrary complex number . The above formula is a generalization of the Newton binomial formula.

9. Compute the Maclaurin series of the function.

(a) $\sqrt{1+x}$.

answer: $\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n$

(b) $\frac{1}{\sqrt{1+x}}$.

answer: $\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^n$

(c) $\frac{1}{\sqrt{1-x^2}}$.

answer: $\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}$

(d) $\arcsin x$.

answer: $\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1}$

Solution. 9.a This problem follows directly from the formula $(1+x)^q = \sum\limits_{n=0}^{\infty}\binom{q}{n}x^n$.

$$\mathrm{Mc}\left(\sqrt{1+x}\right) = \mathrm{Mc}\left((1+x)^{\frac{1}{2}}\right) = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}x^n \quad .$$

Solution. 9.b This problem can be solved by computing the derivative of the preceding problem. However, it is easier to simply apply the generalized Newton Binomial formula.

$$\mathrm{Mc}\left((1+x)^{-\frac{1}{2}}\right) = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^n \quad .$$

Solution. 9.c This problem is solved by replacing x with $-x^2$ in Problem 9.b. To avoid the possible confusion, we carry out the substitution by introducing an intermediate variable y.

$$\begin{aligned}
\mathrm{Mc}\left(\left(1-x^2\right)^{-\frac{1}{2}}\right) &= \mathrm{Mc}\left((1+y)^{-\frac{1}{2}}\right) && \text{Set } y=-x^2 \\
&= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}y^n && \text{Substitute back } y=-x^2 \\
&= \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n} \quad .
\end{aligned}$$

Solution. 9.d We have that $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$, and the Maclaurin series of $\frac{1}{\sqrt{1-x^2}}$ were computed in Problem 9.c. The power series of $\arcsin x$ are therefore obtained via integration.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Mc}(\arcsin x) &= \mathrm{Mc}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x)\right) \\
&= \mathrm{Mc}\left(\frac{1}{\sqrt{1-x^2}}\right) && \text{use Problem } 9.c \\
&= \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n} \\
\mathrm{Mc}(\arcsin x) &= \int\left(\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}\right)\mathrm{d}x \\
&= C+\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\int x^{2n}\mathrm{d}x \\
&= C+\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1} && C=0 \text{ since } \arcsin 0 = 0 \\
&= \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}\frac{x^{2n+1}}{2n+1} \quad .
\end{aligned}$$

10. Find the Taylor series of the function at the indicated point.

(a) $\frac{1}{x^2}$ at $a=-1$.

answer: $1+2(x+1)+3(x+1)^2+\dots = \sum_{n=0}^{\infty}(n+1)(x+1)^n$

(b) $\ln\left(\sqrt{x^2-2x+2}\right)$ at $a=1$.

answer: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{2n}}{2n}$

(c) Write the Taylor series of the function $\ln x$ around $a=2$.

answer: $\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}(x-2)^n$

Solution. 10.b

$$\begin{aligned}
\ln\left(\sqrt{x^2-2x+2}\right) &= \frac{1}{2}\ln\left((x-1)^2+1\right) && \text{use } \ln(1+y)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}, |y|<1 \\
&= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left((x-1)^2\right)^n}{n} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{2n}}{2n} \quad .
\end{aligned}$$

Although the problem does not ask us to do this, we will determine the interval of convergence of the series for exercise. If we use the fact that $\ln(1+y) = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}$ holds for $-1 < y \leq 1$, it follows immediately that the above equality holds for $0 < (x-1)^2 \leq 1$, which holds for $x \in [0,2]$. Let us however compute the interval of convergence without using the aforementioned fact.

Let a_n be the n^{th} term of our series, i.e., let

$$a_n = (-1)^{n+1}\frac{(x-1)^{2n}}{2n} \quad .$$

We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{2n+2}}{(2n+2)}\frac{2n}{(-1)^{n+1}(x-1)^{2n}}\right| \\
&= \lim_{n\to\infty}(x-1)^2\frac{n}{n+1} \\
&= (x-1)^2 \quad .
\end{aligned}$$

By the ratio test, the series is divergent for $(x-1)^2 > 1$, i.e., for $|x-1| > 1$, and convergent for $(x-1)^2 < 1$, i.e., for $|x-1| < 1$. The ratio test is inconclusive at only two points: $x-1 = 1$, i.e., $x = 2$ and $x-1 = -1$, i.e., $x = 0$. At both points the series becomes $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}}{2n}$ and the series is convergent at both points by the alternating series test.

Solution. 10.c This solution is similar to the solution of 10.b, but we have written it in a concise fashion suitable for test taking.

Denote Taylor series at a by T_a and recall that the Maclaurin series of are just T_0, the Taylor series at 0.

$$\begin{aligned}
T_2(\ln x) &= T_2(\ln((x-2)+2)) \\
&= T_2\left(\ln\left(2\left(\frac{x-2}{2}+1\right)\right)\right) \\
&= T_2\left(\ln 2 + \ln\left(1+\frac{x-2}{2}\right)\right) \qquad \left|\; T_0(\ln(1+y)) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}y^n}{n}\right. \\
&= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{x-2}{2}\right)}{n} \\
&= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}(x-2)^n \quad .
\end{aligned}$$

11. Find the Taylor series around the indicated point. The answer key has not been proofread, use with caution.
    (a) $\frac{1}{x}$ at $a = 1$.
    answer: $1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots = \sum\limits_{n=0}^{\infty}(-1)^n(x-1)^n$
    (b) $\frac{1}{x^2}$ at $a = 1$.
    answer: $1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots = \sum\limits_{n=0}^{\infty}(n+1)(-1)^n(x-1)^n$
12. **(This problem is of higher difficulty, it will not appear on the quiz.)** Let $f(x)$ be defined as

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Prove that if $R(x)$ is an arbitrary rational function,

$$\lim_{\substack{x\to 0\\ x>0}} R(x)e^{-\frac{1}{x^2}} = 0$$

   (b) Prove that $f(x)$ is differentiable at 0 and $f'(0) = 0$.
   (c) Prove that the Maclaurin series of $f(x)$ are 0 (but $f(x)$ is clearly a non-zero function).