Freecalc
Homework on Lecture 14, reduced material
includes only material that will enter the Test

1. Determine the interval of convergence for the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

   answer: $x \in [1,3)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

   answer: $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$

   answer: $x \in [0.9, 1.1]$

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$

   answer: $x \in (-2, 0]$

   (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

   answer: $x \in (2, 4]$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   answer: converges for all x

   (g) $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n$

   answer: converges for $|x| < 1$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

   answer: converges for $|x| \in [-1, 1)$.

   (i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

   . answer: converges for $|x| \in (-1, 1]$

   (j) $\displaystyle\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\displaystyle\binom{q}{n}$ stands for $\displaystyle\frac{q(q-1)\dots(q-n+1)}{n!}$.

   answer: converges for $x \in (-1, 1]$.

   Solution. 1.a. We apply the Ratio Test to get that $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2|$. Therefore the power series converges at least in the interval $x \in (1,3)$. When $x = 3$, the series becomes $\sum\limits_{n=1}^{\infty} \frac{1}{3\sqrt{n+1}}$, which diverges - this can be seen, for example, by comparing to the p-series $\frac{1}{\sqrt{n}}$. When $x = 1$, the series becomes $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n+1}}$, which converges by the Alternating Series Test. Our final answer $x \in [1,3)$.

2. For each of the items below, do the following.

   - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).

- Find the radius of convergence of the series you found in the preceding point.

You are not asked to find the entire interval of convergence, but just the radius. In other words, you only need to find the inside of the interval of convergence but do not need to worry for the endpoints. Nevertheless in the answer key we indicate the entire interval of convergence - including the endpoints.

The fact that the value of the series at the endpoints, whenever convergent, coincides with the value of the function has not been demonstrated so far. Nevertheless that is true - but we shall not show it here.

Please post on piazza if you discover errors in the answer key.

(a) $\dfrac{1}{3-x}$.

answer: $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$
$R = 3$
converges for $x \in (-3, 3)$

(b) $\dfrac{1}{3-2x}$.

answer: $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$
$R = \frac{3}{2}$
converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(c) $\dfrac{1}{2x+3}$.

answer: $\frac{1}{3}\left(1 - \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 - \left(\frac{2x}{3}\right)^3 + \dots\right)$
$= \sum_{0}^{\infty} \frac{(-1)^n}{3} \left(\frac{2}{3}\right)^n x^n$
$R = \frac{3}{2}$
converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(d) $\dfrac{1}{1+x^2}$.

answer: $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
$R = 1$
converges for $x \in (-1, 1)$

(e) $\dfrac{1}{1-2x^2}$.

answer: $\sum_{n=0}^{\infty} 2^n x^{2n}$
$R = \frac{1}{2}$
converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(f) $\dfrac{1}{x^2-1}$.

answer: $-\sum_{n=0}^{\infty} x^{2n}$
$R = 1$
converges for $x \in (-1, 1)$

(g) $\dfrac{\frac{1}{2}}{x-1} - \dfrac{\frac{1}{2}}{x+1}$.

answer: same as 2.f

(h) $\dfrac{1}{(1-x)^2}$.

answer: $1 + 2x + 3x^2 + 4x^3 + \dots$
$= \sum_{n=0}^{\infty} (n+1) x^n$
$R = 1$
converges for $x \in (-1, 1)$

(i) $\dfrac{1}{(1-x)^3}$.

answer: $\frac{1}{2}\left(2 + 6x + 12x + \dots + n(n-1)x^{n-2} + \dots\right)$
$= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$
$R = 1$
converges for $x \in (-1, 1)$

(j) $\ln(1+x)$.

answer: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
$R = 1$
converges for $x \in (-1, 1]$

(k) $\ln(1-x)$.

answer: $-\sum_{n=1}^{\infty} \frac{x^n}{n}$
$R = 1$
converges for $x \in [-1, 1)$

(l) $\ln(1-3x)$.

answer: $-\sum_{n=0}^{\infty} \frac{3^n x^n}{n}$
$R = \frac{1}{3}$
converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right]$

(m) $\ln(1-3x^2)$.

answer: $-\sum_{n=1}^{\infty} \frac{3^n x^{2n}}{n}$
$R = \frac{1}{\sqrt{3}}$
converges for $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(n) $\ln(3-2x^2)$.

answer: $\ln 3 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n}}{n}$
$R = \sqrt{\frac{2}{3}}$
converges for $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(o) $x\ln(3-2x^2)$.

answer: $x \ln 3 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n+1}}{n}$
$R = \sqrt{\frac{2}{3}}$
converges for $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(p) $\arctan x$.

answer: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$R = 1$
converges for $x \in (-1, 1]$

(q) $\arctan(2x)$.

answer: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$
$R = \frac{1}{2}$
converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$

(r) $\arctan\left(2x^2\right)$.

answer: $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{4n+2}}{2n+1}$
$R = \frac{1}{\sqrt{2}}$
converges for $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Solution. 2.k

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(1-x)\right) &= \frac{-1}{1-x} && \text{expand as geometric series for } |x|<1\\
&= -\left(1+x+x^2+x^3+\dots\right) && \text{Integrate indefinitely, } |x|<1\\
\ln(1-x) &= -\int \left(1+x+x^2+x^3+\dots\right)\mathrm{d}x && \text{For power series, integral of infinite sum equals infinite sum of integrals inside the convergence radius}\\
&= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)+C && \text{To find } C \text{ set } x=0\\
0=\ln 1 &= -0+C=C\\
\ln(1-x) &= -\left(x+\tfrac{x^2}{2}+\tfrac{x^3}{3}+\dots\right)\\
&= -\sum_{n=1}^{\infty}\frac{x^n}{n} \quad .
\end{aligned}$$

The radius of convergence of the geometric series $1+x+x^2+\dots$ is 1. Since the series for $\ln(1-x)$ is obtained from the geometric series via integration, its radius of convergence is again 1.

We note that the interval of convergence for the series $-\sum_{n=1}^{\infty}\frac{x^n}{n}$ is $[-1,1)$ - the series is convergent at $x=-1$ by the alternating series test and divergent at $x=1$ (at $x=1$ the series is minus the harmonic series). This shows that integration of power series can change convergence at the endpoints of the interval of convergence.

Solution. 2.n. We solve this problem by reducing it to Problem 2.k, which asserts the power series expansion $\ln(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n}$ for $|y|<1$.

$$\begin{aligned}
\ln\left(3-2x^2\right) &= \ln\left(3\left(1-\frac{2}{3}x^2\right)\right)\\
&= \ln 3+\ln\left(1-\frac{2}{3}x^2\right) && \text{Set } y=\tfrac{2}{3}x^2\\
&= \ln 3+\ln(1-y)\\
&= \ln 3-\sum_{n=1}^{\infty}\frac{y^n}{n} && \ln(1-y)=-\textstyle\sum_{n=1}^{\infty}\frac{y^n}{n} \text{ for } |y|<1; \text{ above does not hold for } |y|>1; \text{ above may (not) hold for } y=\pm 1\\
&= \ln 3-\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{x^{2n}}{n} \quad . && \text{Substituted back } y=\tfrac{2}{3}x^2 \quad .
\end{aligned}$$

As indicated above, the equality $\ln(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n}$ holds for $|y|<1$ and fails for $|y|>1$ (for $|y|>1$ the series $\sum_{n=1}^{\infty}\frac{y^n}{n}$ diverges). Therefore interval of convergence is given by

$$\begin{aligned}
|y| &< 1 && \text{use } y=\tfrac{2}{3}x^2\\
\left|\tfrac{2}{3}x^2\right| &< 1\\
\left|x^2\right| &< \tfrac{3}{2}\\
|x| &< \sqrt{\tfrac{3}{2}},
\end{aligned}$$

i.e., the radius of convergence is $R=\sqrt{\frac{3}{2}}$.