Dear students, below you can find the approximate problem types that will appear on the test. The problems are not new, you have already been assigned them in the various homeworks. This file is simply a test prepartaion problem collection for your convenience.

1. Compute the limits. The answer key has not been fully proofread, use with caution.

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$. answer: 1

(b) $\lim\limits_{x\to 0} \dfrac{x}{\ln(1+x)}$. answer: 1

(c) $\lim\limits_{x\to 0} \dfrac{x^2}{x-\ln(1+x)}$. answer: 2

(d) $\lim\limits_{x\to 0} \dfrac{x^2}{\sin x \ln(1+x)}$. answer: 1

(e) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{(\ln(1+x))^2}$. answer: 1

(f) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin x \ln(1+x)}$. answer: $-\frac{1}{2}$

(g) $\lim\limits_{x\to 0} \dfrac{\arctan x - x}{x^3}$. answer: $-\frac{1}{3}$

(h) $\lim\limits_{x\to 0} \dfrac{\arcsin x - x}{x^3}$. answer: $\frac{1}{6}$

(i) $\lim\limits_{x\to 1} \dfrac{x}{x-1} - \dfrac{1}{\ln x}$. answer: $\frac{1}{2}$

(j) $\lim\limits_{x\to 0} \dfrac{\cos(nx)-\cos(mx)}{x^2}$. answer: $\frac{m^2-n^2}{2}$

(k) (Optional) $\lim\limits_{x\to 0} \dfrac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$. answer: $\frac{3}{40}$

2. Express the sum of the series as a rational number.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$ answer: $\frac{13}{6}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5^n}{10^n}$ answer: $\frac{13}{4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n-3^n}{7^n}$ answer: $\frac{7}{4}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}+7^{n-1}}{21^n}$ answer: $\frac{4}{7}$

Solution. 2.a

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n} &= \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n \\
&= \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n + \frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n \qquad \left|\ \begin{array}{l}\text{Use geometric series sum f-la:}\\ \sum\limits_{n=0}^{\infty} r^n = \frac{1}{1-r},\\ \text{provided } |r|<1\end{array}\right. \\
&= \frac{2}{5}\frac{1}{\left(1-\frac{2}{5}\right)} + \frac{3}{5}\frac{1}{\left(1-\frac{3}{5}\right)} \\
&= \frac{13}{6} .
\end{aligned}$$

Solution. 2.b

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n+5^n}{10^n} &= \sum_{n=0}^{\infty}\left(\frac{1}{5^n}+\frac{1}{2^n}\right) \qquad \left|\ \text{use } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \text{ for } |r|<1\right. \\
&= \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{5}} \\
&= \frac{13}{4} .
\end{aligned}$$

Solution. 2.d

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{3^{n+1}+7^{n-1}}{21^n} &= \sum_{n=1}^{\infty}\left(3\frac{3^n}{21^n}+\frac{1}{7}\frac{7^n}{21^n}\right)\\
&= 3\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{7}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\\
&= \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{21}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \qquad \Big|\ \text{use } \textstyle\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1\\
&= \frac{3}{7}\frac{1}{\left(1-\frac{1}{7}\right)}+\frac{1}{21}\frac{1}{\left(1-\frac{1}{3}\right)}\\
&= \frac{4}{7}\quad .
\end{aligned}$$

3. The last problem is intended for the students who are interested. Solving it is not required to be prepared for the test. Use integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n+1}$.

answer: divergent

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2+n^3}$.

answer: convergent

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3}{3n^5+n}$

answer: convergent, can use limit comparison test

(d) $\displaystyle\sum_{2}^{\infty}\frac{1}{n\ln n}$

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(2n+1)\ln(n)}$.

answer: divergent

(f) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

answer: convergent, can use integral test

(g) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(2n+1)(\ln(n))^2}$.

answer: convergent

(h) Determine all values of p, q r for which the series

$$\sum_{n=30}^{\infty}\frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$

is convergent.

Solution. 3.d.

$$\begin{aligned}
\int_2^{\infty}\frac{1}{x\ln x}\mathrm{d}x &= \lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}\mathrm{d}x\\
&= \lim_{t\to\infty}\int_2^t\frac{1}{\ln x}\mathrm{d}(\ln x)\\
&= \lim_{t\to\infty}\int_2^t\mathrm{d}(\ln(\ln x))\\
&= \lim_{t\to\infty}\left[\ln(\ln x)\right]_{x=2}^{x=t}\\
&= \lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)\\
&= \infty,
\end{aligned}$$

therefore the integral is divergent (and diverges to $+\infty$).

The function $\frac{1}{x\ln x}$ is decreasing, as for $x>2$, it is the quotient of 1 by increasing positive functions. $\frac{1}{x\ln x}$ tends to 0 as $x\to\infty$, and therefore the integral criterion implies that $\sum_{2}^{\infty}\frac{1}{n\ln n}$ is divergent.

Solution. 3.e

The integral criterion appears to be of little help: the improper integral $\int\frac{1}{(2x+1)\ln x}\mathrm{d}x$ cannot be integrated algebraically with any of the techniques we have studied so far. Therefore it makes sense to try to solve this problem using a comparison test.

The "dominant term"[1] of the denominator of $\frac{1}{(2n+1)\ln n}=\frac{1}{2n\ln n+\ln n}$ is $2n\ln n$. Therefore it makes sense to compare - or limit-compare - with $\frac{1}{n\ln n}$.

[1]since we do not speak of rational functions, here the expression "dominant term" is used informally

We will use the Limit Comparison Test for the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \frac{1}{(2n+1)\ln n}$ and $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$. Both a_n and b_n are positive (for $n > 2$) and therefore the Limit Comparison Test applies.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(2n+1)\ln n}}{\frac{1}{n\ln n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} \neq 0$, the Limit Comparison Test implies that the series $\sum_{n=2}^{\infty} a_n$ has same convergence/divergence properties as the series $\sum_{n=2}^{\infty} b_n$. In Problem 3.d we demonstrated that the series $\sum\limits_{n=2}^{\infty} b_n$ is divergent; therefore the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \frac{1}{(2n+1)\ln n}$ is divergent as well.

4. Determine the interval of convergence for the following power series.

(a) $\sum\limits_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

answer: $x \in [1,3)$

(b) $\sum\limits_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

answer: $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$

(c) $\sum\limits_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$

answer: $x \in [0.9, 1.1]$

(d) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$

answer: $x \in (-2, 0]$

(e) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

answer: $x \in (2, 4]$

(f) $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$

answer: converges for all x

(g) $\sum\limits_{n=0}^{\infty} (n+1)x^n$

answer: converges for $|x| < 1$

(h) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$

answer: converges for $|x| \in [-1, 1)$.

(i) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

. answer: converges for $|x| \in (-1, 1]$

(j) $\sum\limits_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\binom{q}{n}$ stands for $\frac{q(q-1)\dots(q-n+1)}{n!}$.

answer: converges for $x \in (-1, 1]$.

Solution. 4.a. We apply the Ratio Test to get that $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-2|$. Therefore the power series converges at least in the interval $x \in (1,3)$. When $x = 3$, the series becomes $\sum\limits_{n=1}^{\infty} \frac{1}{3\sqrt{n+1}}$, which diverges - this can be seen, for example, by comparing to the p-series $\frac{1}{\sqrt{n}}$. When $x = 1$, the series becomes $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n+1}}$, which converges by the Alternating Series Test. Our final answer $x \in [1,3)$.

5. Find whether the series is convergent or divergent using an appropriate test.

(a) $\sum\limits_{n=1}^{\infty} (-1)^n \ln n$.

(b) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$.

Solution. 5.a. $\lim\limits_{n\to\infty} (-1)^n \ln n$ does not exist and therefore the sum is not convergent.

Solution. 5.b. For $n > 2$, we have that $\ln n$ is a positive increasing function and therefore $\frac{1}{\ln n}$ is a decreasing positive function. Furthermore $\lim\limits_{n\to\infty} \frac{1}{\ln n} = 0$. Therefore the series is convergent by the alternating series test.

6. For each of the items below, do the following.

- Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).

- Find the radius of convergence of the series you found in the preceding point.

You are not asked to find the entire interval of convergence, but just the radius. In other words, you only need to find the inside of the interval of convergence but do not need to worry for the endpoints. Nevertheless in the answer key we indicate the entire interval of convergence - including the endpoints.

The fact that the value of the series at the endpoints, whenever convergent, coincides with the value of the function has not been demonstrated so far. Nevertheless that is true - but we shall not show it here.

Please post on piazza if you discover errors in the answer key.

(a) $\dfrac{1}{3-x}$.

answer: $\sum\limits_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$
$R = 3$
converges for $x \in (-3, 3)$

(b) $\dfrac{1}{3-2x}$.

answer: $\sum\limits_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$
$R = \frac{3}{2}$
converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(c) $\dfrac{1}{2x+3}$.

answer: $\frac{1}{3}\left(1 - \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 - \left(\frac{2x}{3}\right)^3 + \dots\right)$
$= \sum\limits_{0}^{\infty} \frac{(-1)^n}{3} \left(\frac{2}{3}\right)^n x^n$
$R = \frac{3}{2}$
converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(d) $\dfrac{1}{1+x^2}$.

answer: $\sum\limits_{n=0}^{\infty} (-1)^n x^{2n}$
$R = 1$
converges for $x \in (-1, 1)$

(e) $\dfrac{1}{1-2x^2}$.

answer: $\sum\limits_{n=0}^{\infty} 2^n x^{2n}$
$R = \frac{1}{2}$
converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(f) $\dfrac{1}{x^2-1}$.

answer: $-\sum\limits_{n=0}^{\infty} x^{2n}$
$R = 1$
converges for $x \in (-1, 1)$

(g) $\dfrac{\frac{1}{2}}{x-1} - \dfrac{\frac{1}{2}}{x+1}$.

answer: same as 6.f

(h) $\dfrac{1}{(1-x)^2}$.

answer: $1 + 2x + 3x^2 + 4x^3 + \dots$
$= \sum\limits_{n=0}^{\infty} (n+1) x^n$
$R = 1$
converges for $x \in (-1, 1)$

(i) $\dfrac{1}{(1-x)^3}$.

answer: $\frac{1}{2}\left(2 + 6x + 12x + \dots + n(n-1)x^{n-2} + \dots\right)$
$= \sum\limits_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$
$R = 1$
converges for $x \in (-1, 1)$

(j) $\ln(1+x)$.

answer: $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
$R = 1$
converges for $x \in (-1, 1]$

(k) $\ln(1-x)$.

answer: $-\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$
$R = 1$
converges for $x \in [-1, 1)$

(l) $\ln(1-3x)$.

answer: $-\sum\limits_{n=0}^{\infty} \frac{3^n x^n}{n}$
$R = \frac{1}{3}$
converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right]$

(m) $\ln(1-3x^2)$.

answer: $-\sum\limits_{n=1}^{\infty} \frac{3^n x^{2n}}{n}$
$R = \frac{1}{\sqrt{3}}$
converges for $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(n) $\ln(3-2x^2)$.

answer: $\ln 3 - \sum\limits_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n}}{n}$
$R = \sqrt{\frac{2}{3}}$
converges for $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(o) $x\ln(3-2x^2)$.

answer: $x \ln 3 - \sum\limits_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \frac{x^{2n+1}}{n}$
$R = \sqrt{\frac{2}{3}}$
converges for $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(p) $\arctan x$.

answer: $\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
$R = 1$
converges for $x \in (-1, 1]$

(q) $\arctan(2x)$.

answer: $\sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$
$R = \frac{1}{2}$
converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$

(r) $\arctan\left(2x^2\right)$.

answer: $\sum\limits_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{4n+2}}{2n+1}$
$R = \frac{1}{\sqrt{2}}$
converges for $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Solution. 6.k

$$\begin{array}{rcll}
\frac{\mathrm{d}}{\mathrm{d}x}(\ln(1-x)) & = & \frac{-1}{1-x} & \text{expand as geometric series} \\
 & & & \text{for } |x|<1 \\
 & = & -\left(1+x+x^2+x^3+\dots\right) & \text{Integrate indefinitely, } |x|<1 \\
 & & & \text{For power series,} \\
 & & & \text{integral of infinite} \\
\ln(1-x) & = & -\int\left(1+x+x^2+x^3+\dots\right)\mathrm{d}x & \text{sum equals} \\
 & & & \text{infinite sum of integrals} \\
 & & & \text{inside the convergence radius} \\
 & = & -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)+C & \text{To find } C \text{ set } x=0 \\
0=\ln 1 & = & -0+C=C & \\
\ln(1-x) & = & -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right) & \\
 & = & -\sum_{n=1}^{\infty}\frac{x^n}{n} \quad . &
\end{array}$$

The radius of convergence of the geometric series $1+x+x^2+\dots$ is 1. Since the series for $\ln(1-x)$ is obtained from the geometric series via integration, its radius of convergence is again 1.

We note that the interval of convergence for the series $-\sum_{n=1}^{\infty}\frac{x^n}{n}$ is $[-1,1)$ - the series is convergent at $x=-1$ by the alternating series test and divergent at $x=1$ (at $x=1$ the series is minus the harmonic series). This shows that integration of power series can change convergence at the endpoints of the interval of convergence.

Solution. 6.n. We solve this problem by reducing it to Problem 6.k, which asserts the power series expansion $\ln(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n}$ for $|y|<1$.

$$\begin{array}{rcll}
\ln\left(3-2x^2\right) & = & \ln\left(3\left(1-\frac{2}{3}x^2\right)\right) & \\
 & = & \ln 3+\ln\left(1-\frac{2}{3}x^2\right) & \text{Set } y=\frac{2}{3}x^2 \\
 & = & \ln 3+\ln(1-y) & \\
 & & & \ln(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n} \text{ for } |y|<1 \\
 & = & \ln 3-\sum_{n=1}^{\infty}\frac{y^n}{n} & \text{above does not hold for } |y|>1 \\
 & & & \text{above may (not) hold for } y=\pm 1 \\
 & = & \ln 3-\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{x^{2n}}{n} \quad . & \text{Substituted back } y=\frac{2}{3}x^2 \quad .
\end{array}$$

As indicated above, the equality $\ln(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n}$ holds for $|y|<1$ and fails for $|y|>1$ (for $|y|>1$ the series $\sum_{n=1}^{\infty}\frac{y^n}{n}$ diverges). Therefore interval of convergence is given by

$$\begin{array}{rcll}
|y| & < & 1 & \text{use } y=\frac{2}{3}x^2 \\
\left|\frac{2}{3}x^2\right| & < & 1 & \\
\left|x^2\right| & < & \frac{3}{2} & \\
|x| & < & \sqrt{\frac{3}{2}}, &
\end{array}$$

i.e., the radius of convergence is $R=\sqrt{\frac{3}{2}}$.

Solution. 6.h

$$
\begin{array}{rcll}
\dfrac{1}{1-x} & = & 1+x+x^2+x^3+\dots & \Big|\ \text{geometric series }, |x|<1 \\
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x}\right) & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1+x+x^2+x^3+\dots\right) & \Big|\ \text{apply } \frac{\mathrm{d}}{\mathrm{d}x} \\
-\dfrac{(1-x)'}{(1-x)^2} = \dfrac{1}{(1-x)^2} & = & 1+2x+3x^2+\dots & \\
\dfrac{1}{(1-x)^2} & = & \displaystyle\sum_{n=0}^{\infty}(n+1)x^n & \Big|\ \text{rewrite in } \sum \text{ notation.}
\end{array}
$$

The radius of convergence of the geometric series is 1. Since differentiating does not change the radius of convergence, the radius of convergence of $\frac{1}{(1-x)^2} = \sum\limits_{n=0}^{\infty}(n+1)x^n$ is $R = 1$.

The problem does not ask us to determine the interval of convergence, however let us do it for exercise. The endpoints of the interval of convergence are -1 and 1. The series is divergent for of them: indeed, at $x = -1$ the series becomes $\sum\limits_{n=0}^{n}(-1)^n(n+1)$ and at $x = 1$ the series becomes $\sum\limits_{n=0}^{n}(n+1)$. Both of these series are divergent as their terms do not tend to 0 as n tends to infinity. Thus the interval of convergence is $x \in (-1, 1)$.