

Math 141

Lecture 15

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University of Massachusetts Boston

Spring 2015

Outline

1 Polar Coordinates

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1 Polar Coordinates

2 Complex numbers

Polar Coordinates

- The polar coordinate system is an alternative to the Cartesian coordinate system.

Polar Coordinates

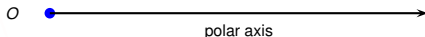
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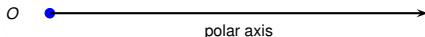


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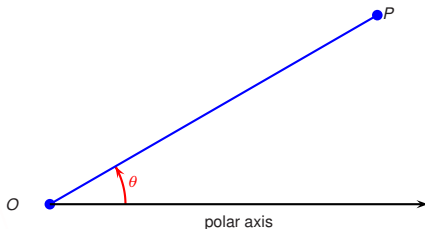
• P

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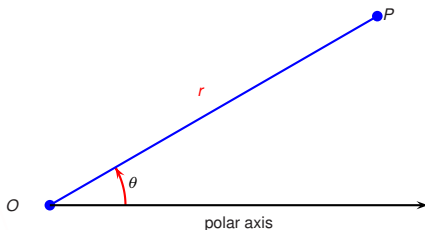
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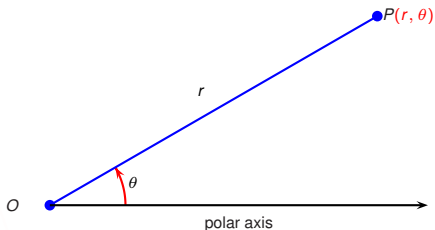
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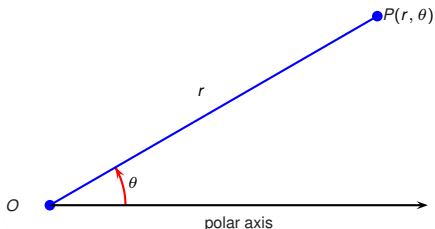
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- Let P be a point in the plane.
- Let θ denote the angle between the polar axis and the line OP .
- Let r denote the length of the segment OP .
- Then P is represented by the ordered pair (r, θ) .

Polar Coordinates

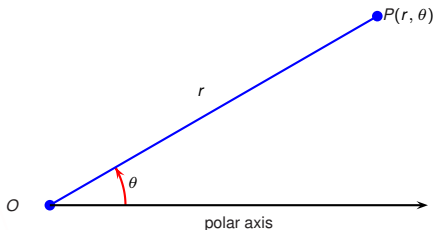
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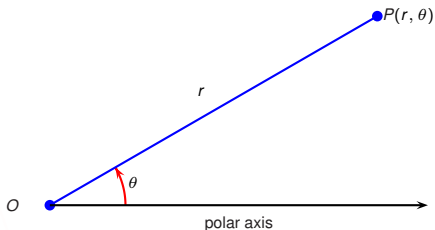
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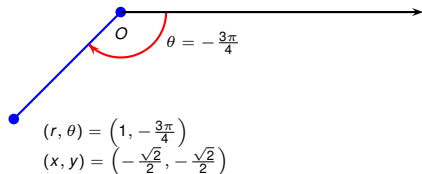
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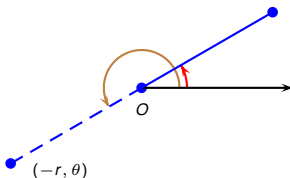
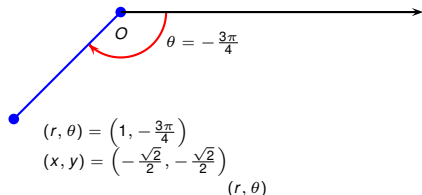
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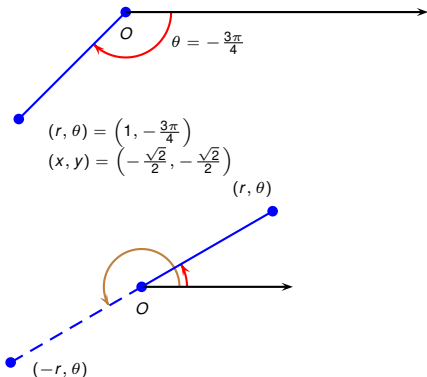
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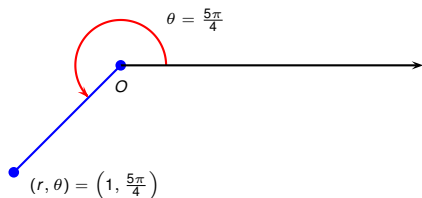


- 1 Positive angles θ are measured in the counterclockwise direction from O . Negative angles are measured in the clockwise direction.
- 2 Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O , but on opposite sides.

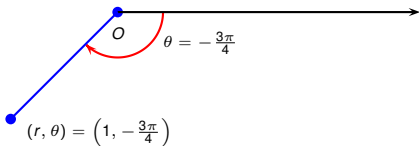
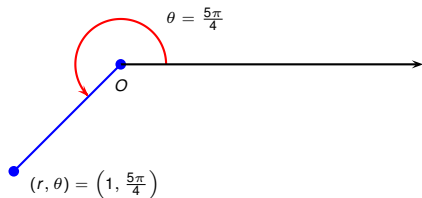
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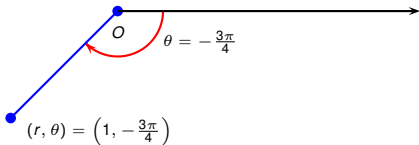
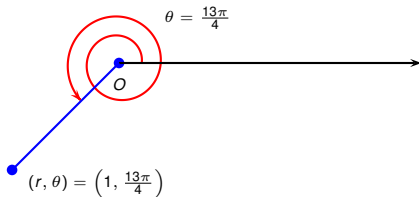
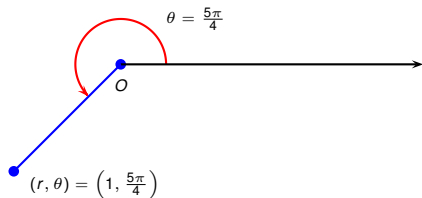
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- 2 Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O , but on opposite sides.
- 3 If $r = 0$, then $(0, \theta)$ represents O for all values of θ .



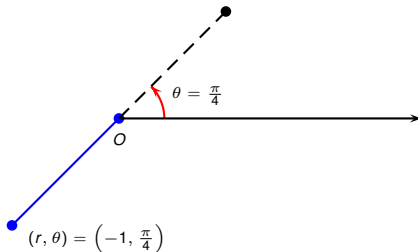
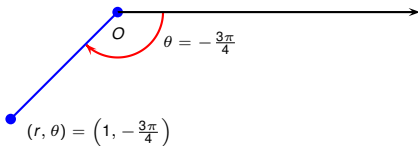
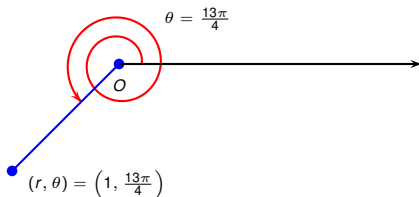
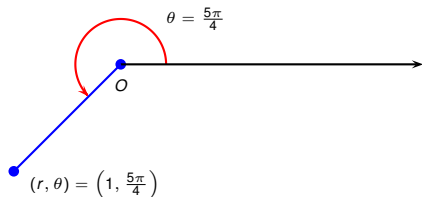
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- There are many ways to represent the same point.
- We could use a negative θ .
- We could go around more than once.
- We could use a negative r .

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Observation

P_1 coincides with P_2 if one of the three mutually exclusive possibilities holds:

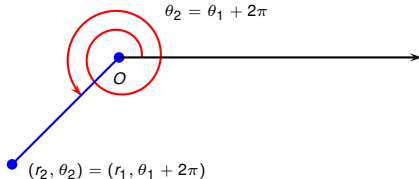
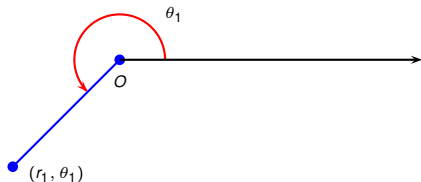
- $r_1 = r_2 \neq 0$ and $\theta_2 = \theta_1 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ and $\theta_2 = \theta_1 + (2k + 1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ and θ is arbitrary.

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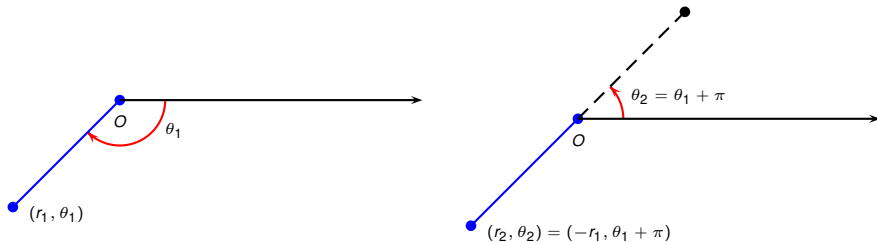


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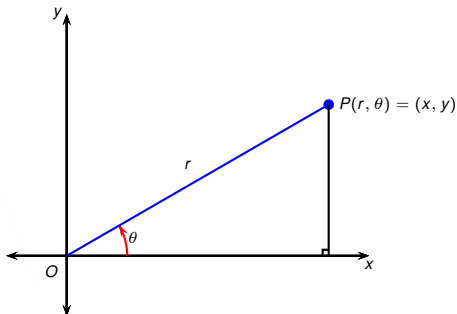
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- How do we go from polar coordinates to Cartesian coordinates?



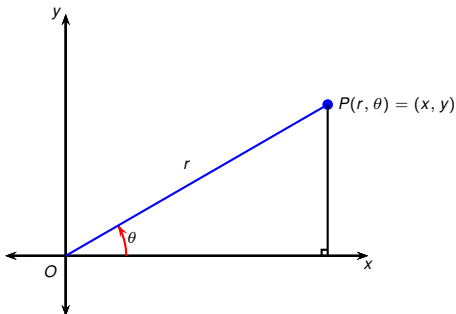
$$x =$$

$$y =$$

$$r =$$

$$\theta =$$

- How do we go from polar coordinates to Cartesian coordinates?
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y) .



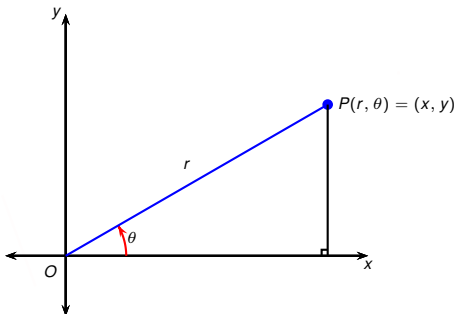
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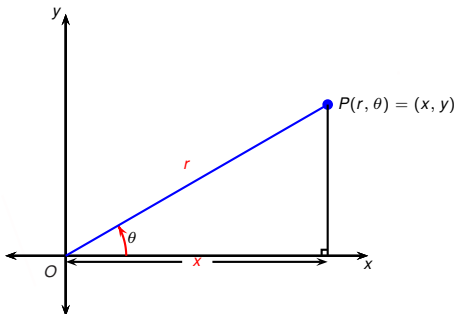
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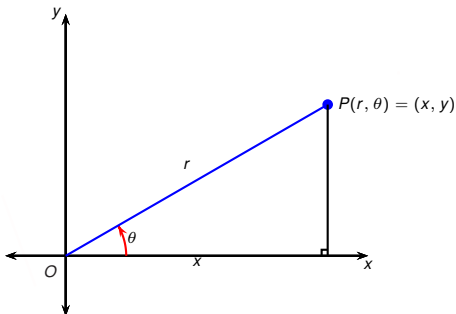
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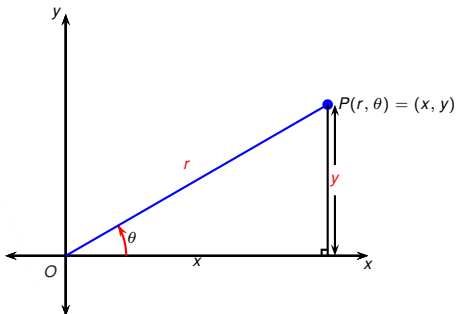
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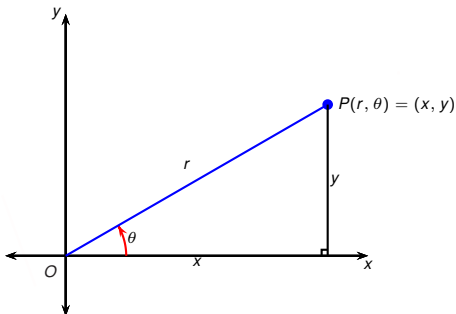
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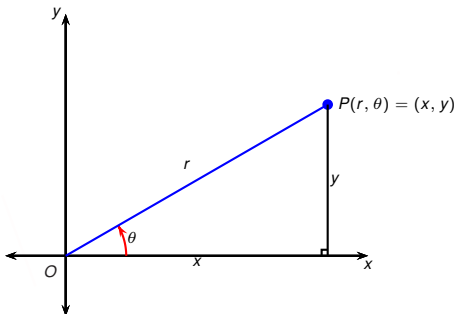
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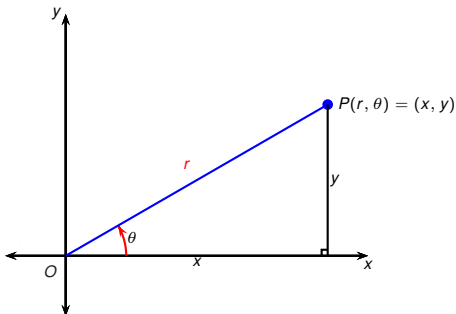
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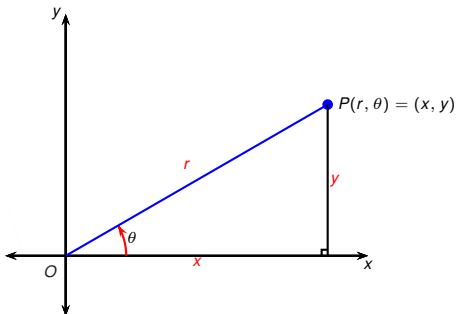
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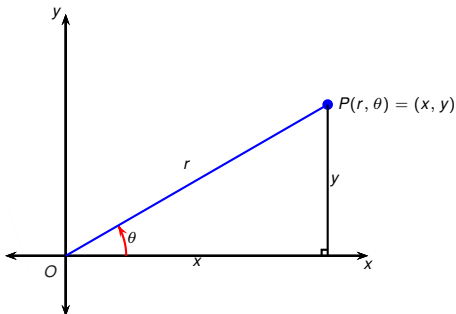
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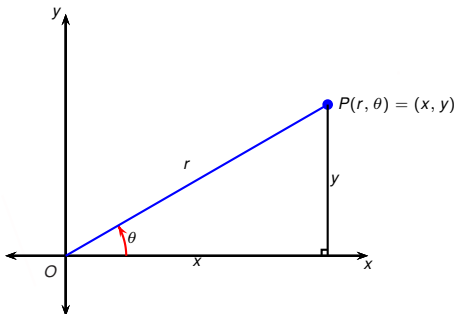
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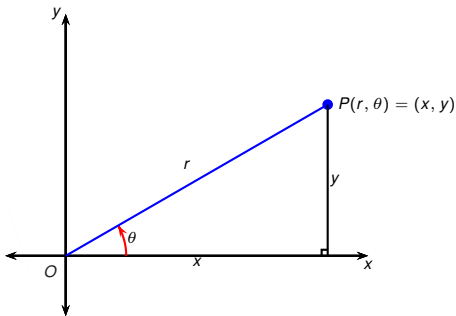
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$$\theta = \arcsin\left(\frac{y}{r}\right) \quad \text{if } x > 0$$

$$= \arccos\left(\frac{x}{r}\right) \quad \text{if } y > 0$$

$$= \arctan\left(\frac{y}{x}\right) \quad \text{if } x > 0$$

Example

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$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \left(\frac{1}{2} \right)$$

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Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

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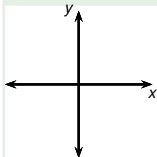
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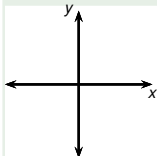
Therefore the point with polar coordinates $(2, \frac{\pi}{3})$ has Cartesian coordinates $(1, \sqrt{3})$.

Example



Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example

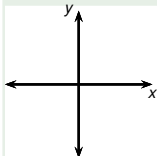


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

$$r = \pm \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$

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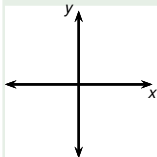
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- Suppose r is positive.

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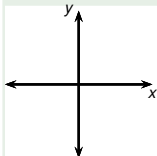
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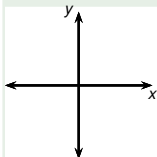
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$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

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Example



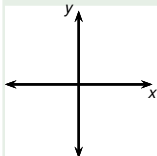
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

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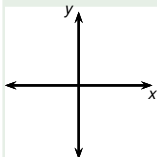
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- Suppose r is positive.
- $\tan \theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.

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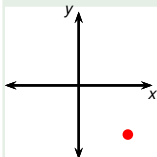
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- Suppose r is positive.
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- $(1, -1)$ is in the fourth quadrant.

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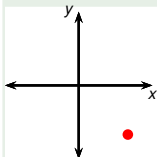
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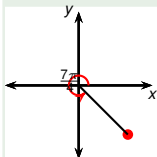
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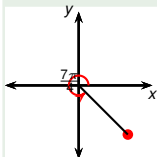
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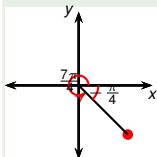
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- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- Therefore one possible representation of $(1, -1)$ in polar coordinates is $(\sqrt{2}, 7\pi/4)$.

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- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
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Complex numbers definition

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The set of complex numbers \mathbb{C} is defined as the set

$$\{a + bi \mid a, b - \text{real numbers}\},$$

where the number i is a number for which

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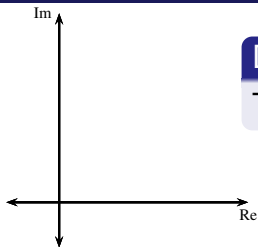
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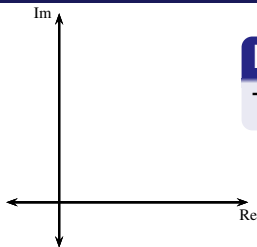
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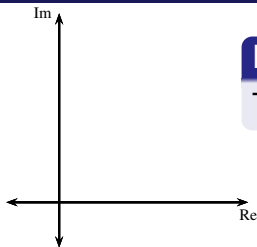
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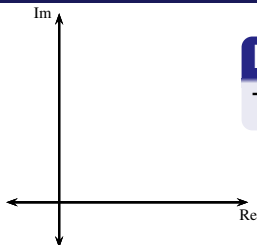
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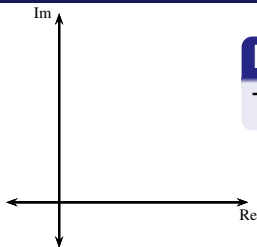


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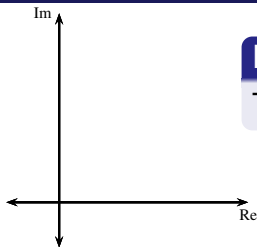
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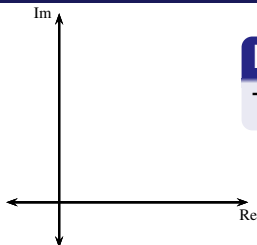
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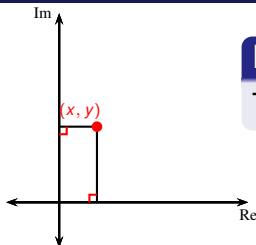
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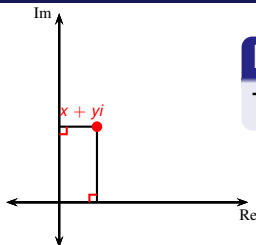
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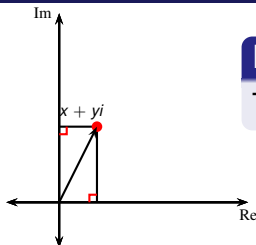
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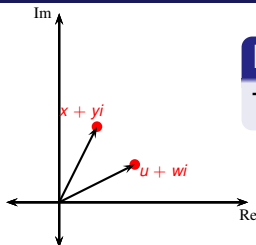
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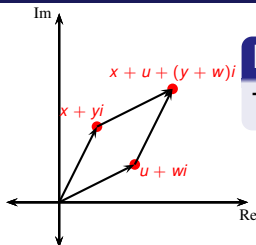
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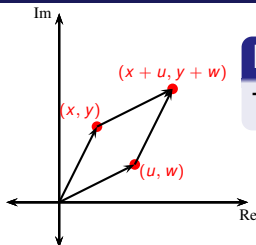
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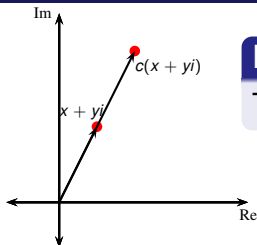
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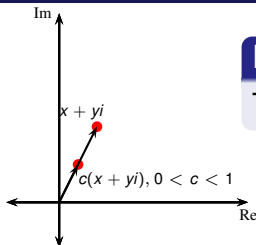
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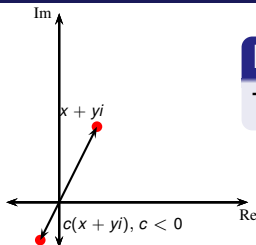
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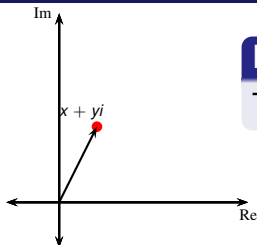
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- The space the complex numbers is referred to as the **complex plane** (sometimes alternatively called the complex line).

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$$u + v =$$

.

Example (Subtraction)

$$u - v =$$

.

Example (Multiplication)

$$u \cdot v =$$

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$$u + v = (2 + 3i) + (5 - 7i) = ?$$

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$$u + v = (2 + 3i) + (5 - 7i) = (2 + 5) + (3 - 7)i = 7 - 4i.$$

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$$\begin{aligned} u \cdot v &= (2 + 3i) \cdot (5 - 7i) \\ &= 2 \cdot 5 + 2 \cdot (-7)i + 3i \cdot 5 + 3i(-7i) \\ &= 10 - 14i + 15i - 21i^2 \end{aligned}$$

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$$\begin{aligned} u \cdot v &= (2 + 3i) \cdot (5 - 7i) \\ &= 2 \cdot 5 + 2 \cdot (-7)i + 3i \cdot 5 + 3i(-7i) \\ &= 10 - 14i + 15i - 21i^2 \end{aligned}$$

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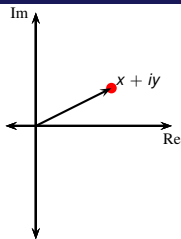
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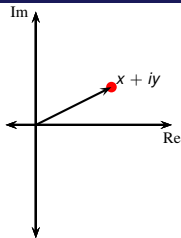
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Let $z = x + iy$ be a complex number.

Definition (Complex conjugation)

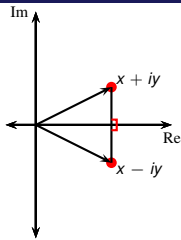
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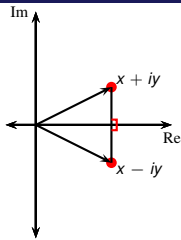


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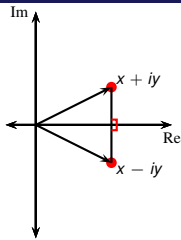
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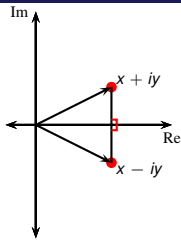
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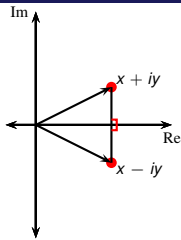
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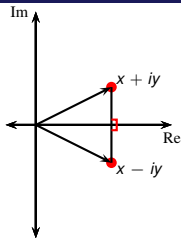
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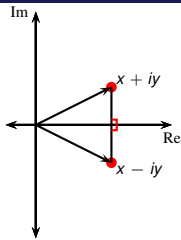
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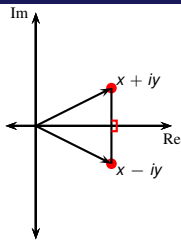
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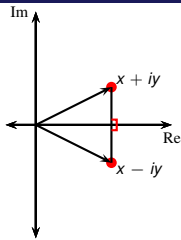
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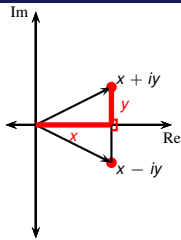
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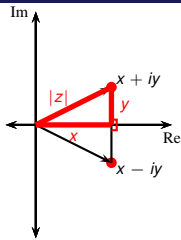
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Theorem (Conjugation preserves $+$, \cdot)

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$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \quad w \neq 0.$$

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Division)

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Multiply and divide
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Multiply and divide
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Let $u = a + bi$, $v = c + di$, $v \neq 0$.

Example (Complex number division)

$$\begin{aligned}
 \frac{u}{v} &= \frac{a + bi}{c + di} \\
 &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\
 &= \frac{(a + bi)(c - di)}{c^2 - (di)^2} \\
 &= \frac{ac - adi + cbi - bdi^2}{c^2 + d^2} \\
 &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\
 &= \frac{ac + bd}{c^2 + d^2} + \frac{(bc - ad)}{c^2 + d^2}i
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Multiply and divide
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Definition (Complex number division)

The quotient $\frac{u}{v}$, $v \neq 0$ is defined via the formula above.

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Proof.



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$$\begin{aligned} e(z)e(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{s=0}^{\infty} \sum_{k=0}^s \frac{z^k w^{s-k}}{k!(s-k)!} \\ &= \sum_{s=0}^{\infty} \sum_{k=0}^s \frac{z^k w^{s-k}}{s!} \frac{s!}{k!(s-k)!} = \sum_{s=0}^{\infty} \frac{(z+w)^s}{s!} = e(z+w). \end{aligned}$$



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Lemma (Newton Binomial formula)

$$(z+w)^s = \sum_{k=0}^s z^k w^{s-k} \frac{s!}{k!(s-k)!}.$$

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Let $z \in \mathbb{R}$. The real exponent e^z is defined as $\lim_{\substack{p \rightarrow x \\ p \text{ is rational}}} e^p$.

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Under Definition I the Maclaurin series of e^z was computed to be

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:



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Rearrange.



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$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

Rearrange. Plug-in $z = ix$.



Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant.

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Lemma

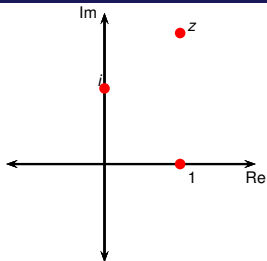
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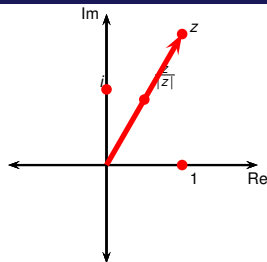
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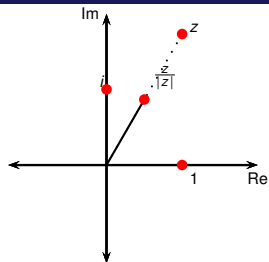
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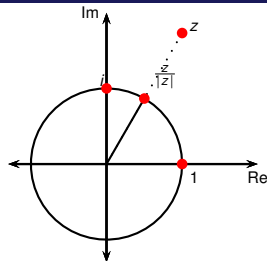
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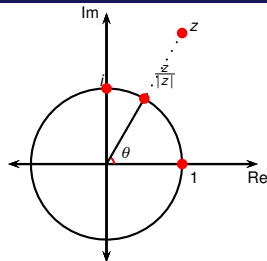
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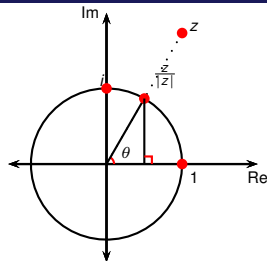
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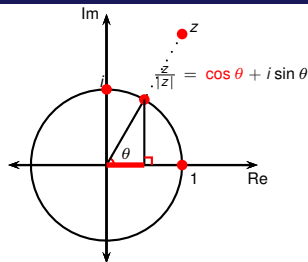
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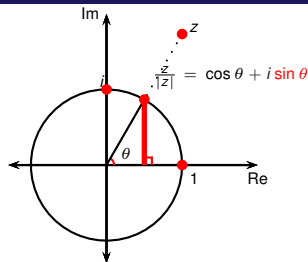
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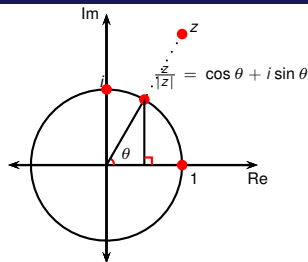
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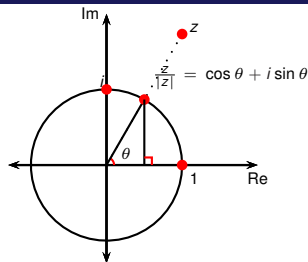
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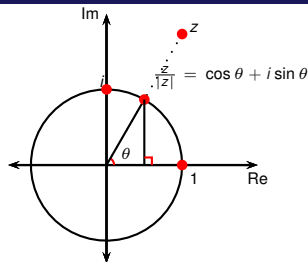
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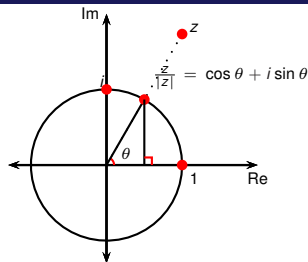
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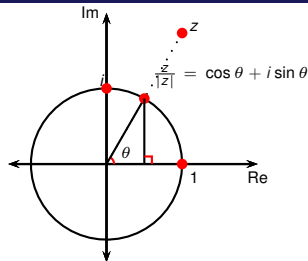
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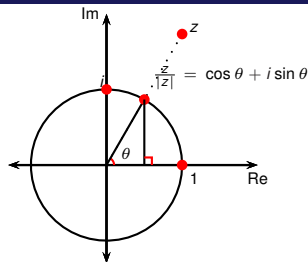
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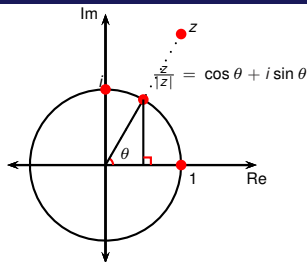
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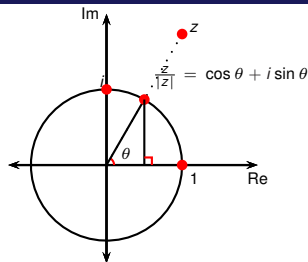
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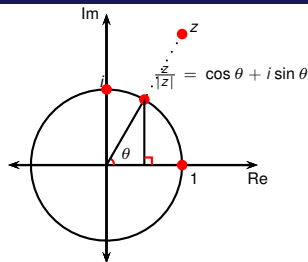
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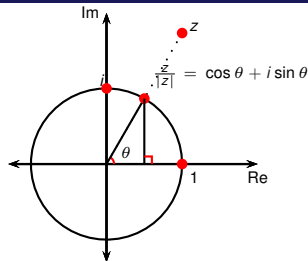
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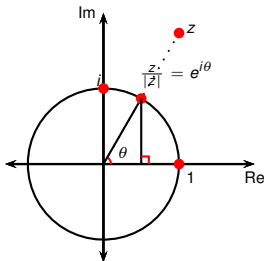
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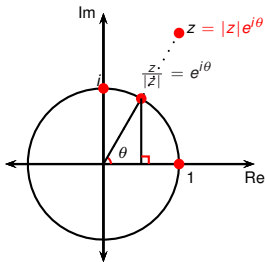
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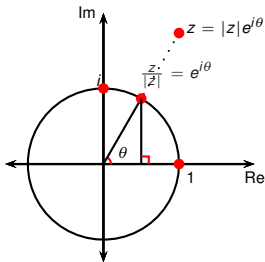


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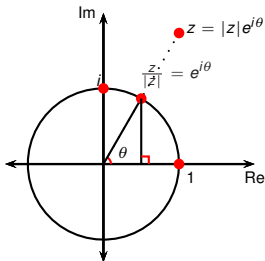
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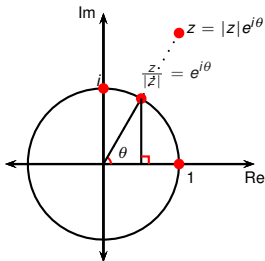
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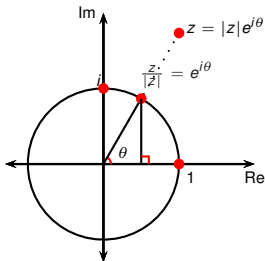
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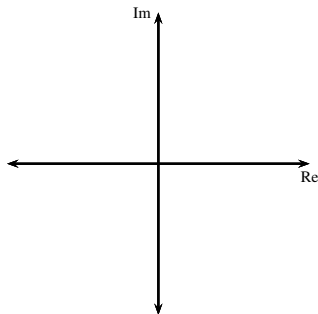
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- One should never write $\theta = \arg z$ without clarifying the choice of argument.

Example

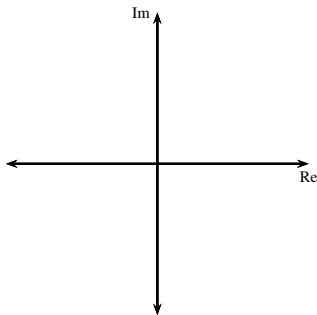


Plot the number z . Write z in polar form, using the principal value of the argument of z (polar angle).

Recall that θ is the principal argument $\Rightarrow \theta \in (-\pi, \pi]$.

z	$ z $	θ	$ z (\cos \theta + i \sin \theta)$
1	1	0	$\cos 0 + i \sin 0$
i	1	$\frac{\pi}{2}$	$\cos \left(\frac{\pi}{2}\right) + i \sin \left(\frac{\pi}{2}\right)$
-1	1	π	$\cos \pi + i \sin \pi$
$-i$	1	$-\frac{\pi}{2}$	$\cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right)$

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z	$ z $	θ	$ z (\cos \theta + i \sin \theta)$
$\frac{1}{2} + \frac{\sqrt{3}}{2}i$	1	$\frac{\pi}{3}$	$\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$
$1 + i$	2	$\frac{\pi}{4}$	$2\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$
$1 - i$	2	$-\frac{\pi}{4}$	$2\left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right)$
$-\sqrt{3} - i$	2	$-\frac{2\pi}{3}$	$2\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)$
$\frac{3}{5} + \frac{4}{5}i$	5	$\arctan\left(\frac{4}{3}\right)$	$5\left(\cos\left(\arctan\left(\frac{4}{3}\right)\right) + i \sin\left(\arctan\left(\frac{4}{3}\right)\right)\right)$

Definition (Real exponent)

Let $\rho \in \mathbb{R}$. The real exponent e^ρ is defined as $\lim_{\substack{p \rightarrow \rho \\ p \text{ is rational}}} e^p$.

- For the duration of this slide, assume Definition I of real exponent.

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Let $\rho, \theta \in \mathbb{R}$. Define the complex exponent $e^{\rho+i\theta}$ via $e^{\rho+i\theta} = e^\rho (\cos \theta + i \sin \theta)$

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- The trig. f-las used above need separate (relatively long) proof.

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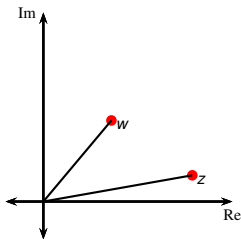
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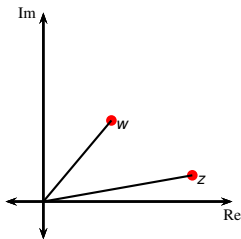
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Geometric interpretation of complex multiplication



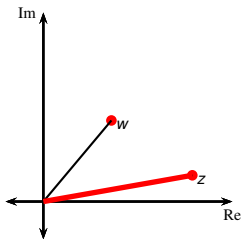
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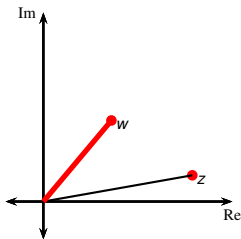
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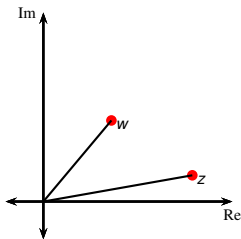
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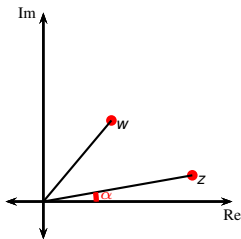


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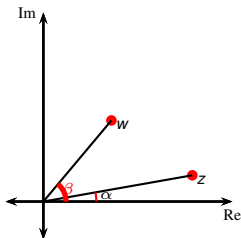


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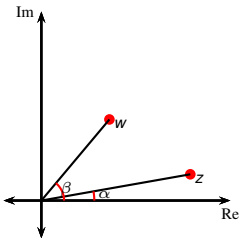


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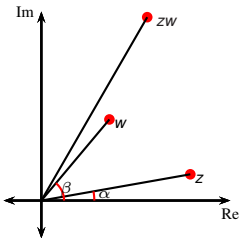


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 &= e^{\rho+\sigma+i(\alpha+\beta)} = |z||w|(\cos(\alpha + \beta) + i \sin(\alpha + \beta)).
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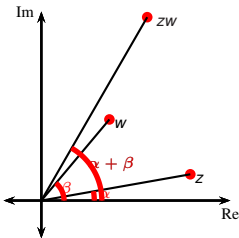


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 &= e^\rho(\cos \alpha + i \sin \alpha)e^\sigma(\cos \beta + i \sin \beta) = e^{\rho+i\alpha}e^{\sigma+i\beta} \\
 &= e^{\rho+\sigma+i(\alpha+\beta)} = |z||w|(\cos(\alpha + \beta) + i \sin(\alpha + \beta)).
 \end{aligned}$$

Geometric interpretation of complex multiplication



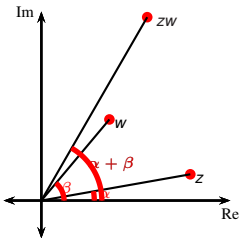
- Let $z, w \neq 0$ and let
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- Let α, β be arguments of z, w .
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Geometric interpretation of complex multiplication



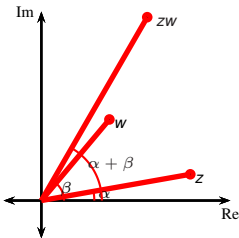
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Geometric interpretation of complex multiplication



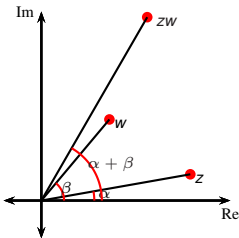
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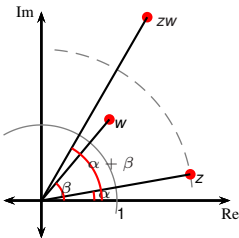
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Geometric interpretation of complex multiplication



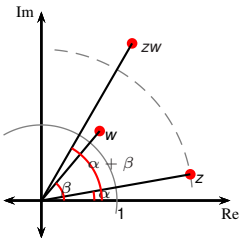
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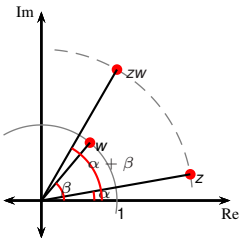
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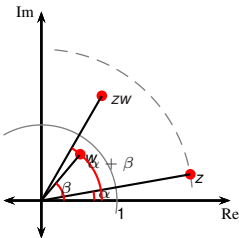
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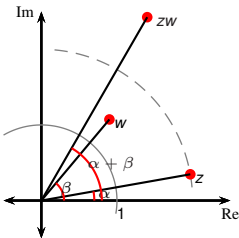
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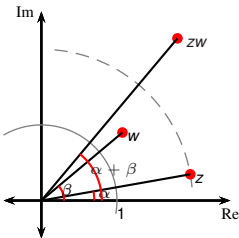
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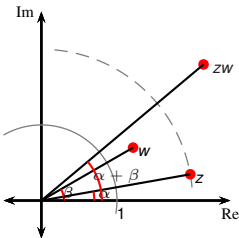
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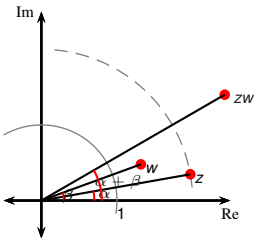
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Theorem (de Moivre's formula)

$$(\cos \alpha + i \sin \alpha)^n = \cos(n\alpha) + i \sin(n\alpha).$$

Proof.



The formula is named after the French mathematician A. de Moivre (1667-1754).

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The formula is named after the French mathematician A. de Moivre (1667-1754).

Polar form $z = |z|(\cos \theta + i \sin \theta)$.

Example

Compute $(\sqrt{3} + i)^{2014}$ and its polar form.

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Example

Compute $(\sqrt{3} + i)^{2014}$ and its polar form.

Write $\sqrt{3} + i$ in polar form: $\sqrt{3} + i = ? (\cos (?) + i \sin (?))$.

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 &= -2^{2014} \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)
 \end{aligned}$$

Polar form $z = |z|(\cos \theta + i \sin \theta)$.

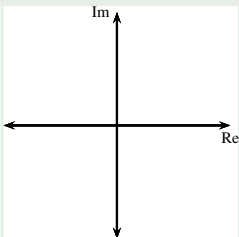
Example

Compute $(\sqrt{3} + i)^{2014}$ and its polar form.

Write $\sqrt{3} + i$ in polar form: $\sqrt{3} + i = 2 \left(\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right) = 2e^{i\frac{\pi}{6}}$.

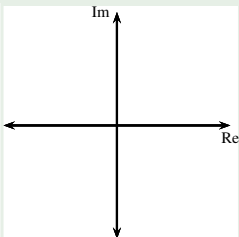
$$\begin{aligned}
 (\sqrt{3} + i)^{2014} &= \left(2e^{i\frac{\pi}{6}} \right)^{2014} \\
 &= 2^{2014} e^{i2014 \cdot \frac{\pi}{6}} \\
 &= 2^{2014} (e^{i(335 + \frac{2}{3})\pi}) \\
 &= 2^{2014} e^{i335\pi} e^{i\frac{2}{3}\pi} \\
 &= 2^{2014} (-1) \left(\cos \left(\frac{2\pi}{3} \right) + i \sin \left(\frac{2\pi}{3} \right) \right) \\
 &= -2^{2014} \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\
 &= 2^{2013} (1 - \sqrt{3}i).
 \end{aligned}$$

Example



Find all complex solutions of the equation $z^4 = 1$.

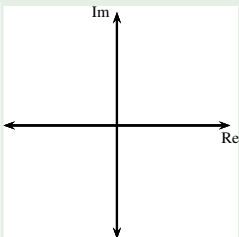
Example



Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$.

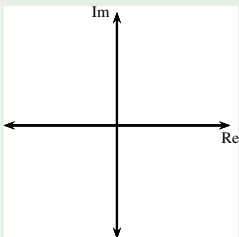
Example



Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$

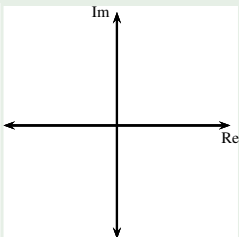
Example



Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

Example

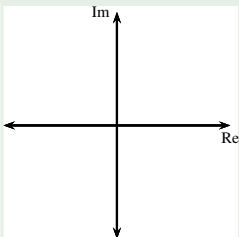


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$.

Example

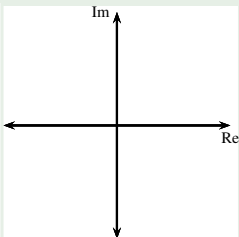


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$

Example

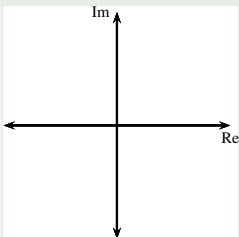


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k -integer.

Example

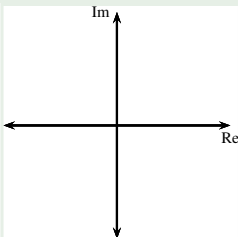


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k -integer. Therefore $\theta = k\frac{\pi}{2}$.

Example

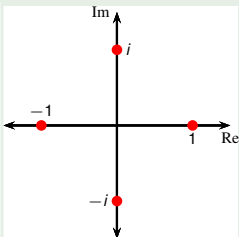


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k -integer. Therefore $\theta = k\frac{\pi}{2}$. Among those values, $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ belong to $(-\pi, \pi]$.

Example

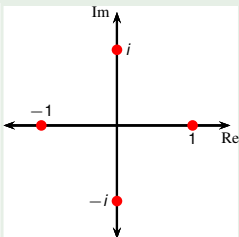


Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k -integer. Therefore $\theta = k\frac{\pi}{2}$. Among those values, $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ belong to $(-\pi, \pi]$. We may discard the other values of θ as do not give rise to new points.

Example



Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos \theta + i \sin \theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i \sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k -integer. Therefore $\theta = k\frac{\pi}{2}$. Among those values, $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ belong to $(-\pi, \pi]$. We may discard the other values of θ as do not give rise to new points.

Therefore the equation $z^4 = 1$ has 4 roots given by

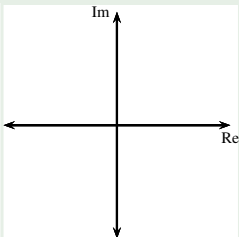
$$z = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i$$

$$z = \cos 0 + i \sin 0 = 1$$

$$z = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i$$

$$z = \cos \pi + i \sin \pi = -1$$

Example



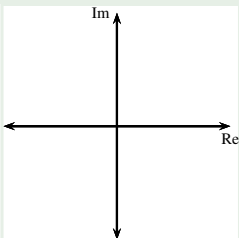
Find all complex numbers z such that $z^3 = i$.

$$\begin{aligned}
 i &= (\cos(\theta) + i \sin(\theta))^3 \\
 \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) &= \cos(3\theta) + i \sin(3\theta) \\
 3\theta &= \frac{\pi}{2} + 2k\pi \\
 \theta &= \frac{\pi}{6} + \frac{2k}{3}\pi
 \end{aligned}$$

de Moivre
Polar form i
 k any integer
 k any integer

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$
 $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$. Our final answer is **to be continued**.

Example

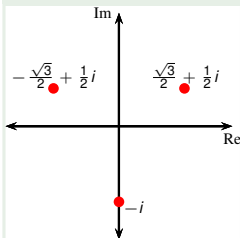


Find all complex numbers z such that $z^3 = i$.
 Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z
 for which $\theta \in (-\pi, \pi]$.

$$\begin{array}{rcl}
 i & = & (\cos(\theta) + i \sin(\theta))^3 \\
 \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) & = & \cos(3\theta) + i \sin(3\theta) \\
 3\theta & = & \frac{\pi}{2} + 2k\pi \\
 \theta & = & \frac{\pi}{6} + \frac{2k}{3}\pi
 \end{array}
 \quad \left| \begin{array}{l} \text{de Moivre} \\ \text{Polar form } i \\ k \text{ any integer} \\ k \text{ any integer} \end{array} \right.$$

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$
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Example



Find all complex numbers z such that $z^3 = i$.

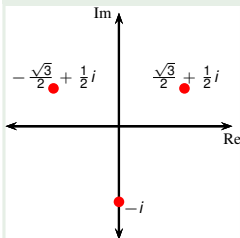
Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z for which $\theta \in (-\pi, \pi]$. We have that

$$1 = |i| = |z^3| = |z|^3.$$

i	$=$	$(\cos(\theta) + i \sin(\theta))^3$	de Moivre Polar form i k any integer k any integer
$\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$	$=$	$\cos(3\theta) + i \sin(3\theta)$	
3θ	$=$	$\frac{\pi}{2} + 2k\pi$	
θ	$=$	$\frac{\pi}{6} + \frac{2k}{3}\pi$	

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$
 $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$. Our final answer is **to be continued**.

Example



Find all complex numbers z such that $z^3 = i$.

Let $z = |z|(\cos \theta + i \sin \theta)$ be the polar form of z for which $\theta \in (-\pi, \pi]$. We have that

$1 = |i| = |z^3| = |z|^3$. Since $|z|$ is a positive real number, $|z|^3 = 1$ implies $|z| = 1$.

$$\begin{array}{rcl}
 i & = & (\cos(\theta) + i \sin(\theta))^3 \\
 \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) & = & \cos(3\theta) + i \sin(3\theta) \\
 3\theta & = & \frac{\pi}{2} + 2k\pi \\
 \theta & = & \frac{\pi}{6} + \frac{2k}{3}\pi
 \end{array}
 \quad \left| \begin{array}{l} \text{de Moivre} \\ \text{Polar form } i \\ k \text{ any integer} \\ k \text{ any integer} \end{array} \right.$$

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$
 $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$. Our final answer is **to be continued**.