Math 141

Lecture 16[material reduced]

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Spring 2015

Outline

1. Curves
   - The Cycloid

Outline

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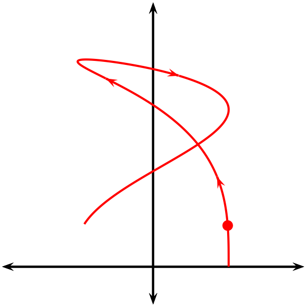
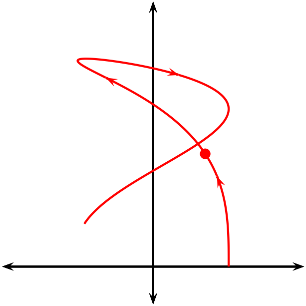
Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.

Curves Defined by Parametric Equations



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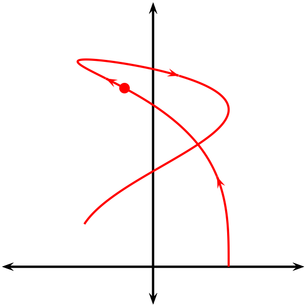

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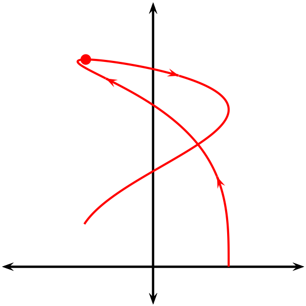

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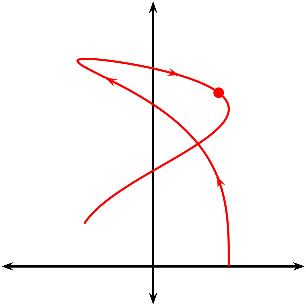

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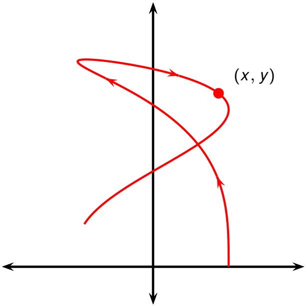
Curves Defined by Parametric Equations



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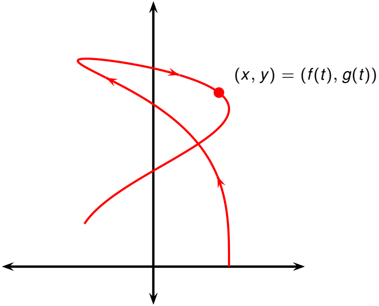
- The x-coordinate and y-coordinate of the particle are some functions of the time t.

Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.
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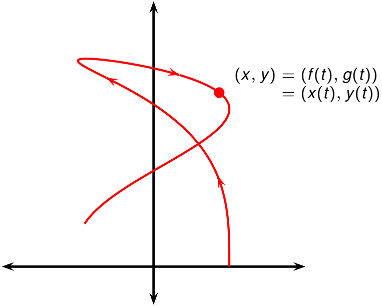
- We can write $x = f(t)$ and $y = g(t)$.

Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.
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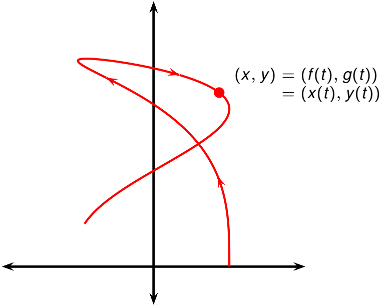
- Less formally, we may directly write $(x, y) = (x(t), y(t))$.

Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.
- The x-coordinate and y-coordinate of the particle are some functions of the time t.
- We can write $x = f(t)$ and $y = g(t)$.
- Less formally, we may directly write $(x, y) = (x(t), y(t))$.
- We say that the equations $\left| \begin{array}{lcl} x & = & f(t) \\ y & = & g(t) \end{array} \right.$ are parametric equations of a parametric curve.
- Note that the curve can't be written as $y = f(x)$: it fails the vertical line test.

Definition (Curve in *n*-dimensional space)

We define an arbitrary *n*-tuple of functions $f_1, \dots, f_n$ on $[a, b]$ to be a *parametric curve* (or simply *curve*). If C is a curve, we write C as:

$$C : \left| \begin{array}{ccc} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

where $x_1, \dots, x_n$ are the labels of the *n*-dimensional coordinate system.

Curves in 2- and 3-dimensional space will be of special interest:

A curve in dimension 2 is given by:

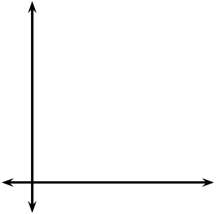
$$C : \left| \begin{array}{ccc} x & = & f(t) \\ y & = & g(t) \end{array} \right. , t \in [a, b] \quad .$$

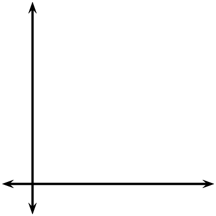
A curve in dimension 3 is given by:

$$C : \left| \begin{array}{ccc} x & = & f(t) \\ y & = & g(t) \\ z & = & h(t) \end{array} \right. , t \in [a, b] \quad .$$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0,1] \qquad \gamma_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0,1]$$





Consider the two parametric curves:

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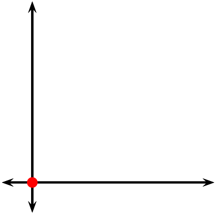

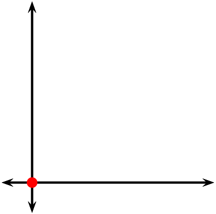

Plug in $t = 0$

Consider the two parametric curves:

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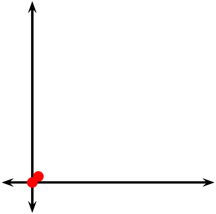
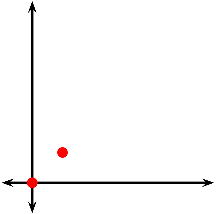
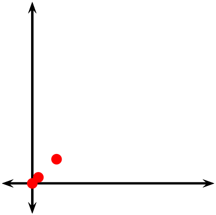


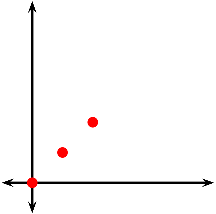


Plug in $t = 0$, $t = 0.2$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{lll} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1] \qquad \gamma_2 : \left| \begin{array}{lll} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

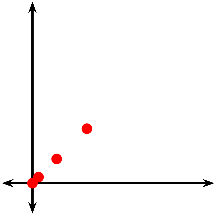


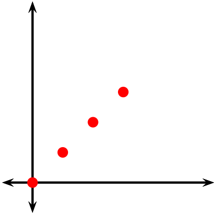


Plug in $t = 0$, $t = 0.2$, $t = 0.4$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{rcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0,1] \qquad \gamma_2 : \left| \begin{array}{rcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0,1]$$

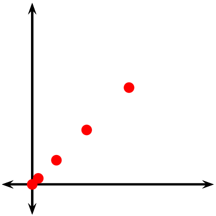


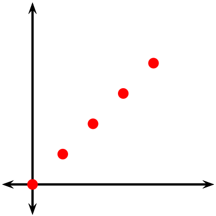


Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{ccc} x & = & t^2 \\ y & = & t^2 \end{array} \right., t \in [0, 1] \qquad \gamma_2 : \left| \begin{array}{ccc} x & = & t \\ y & = & t \end{array} \right., t \in [0, 1]$$





Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$

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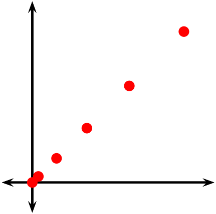

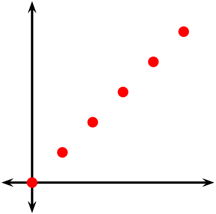

Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

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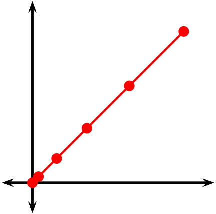

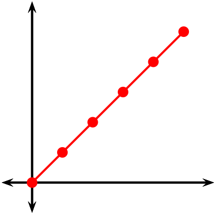

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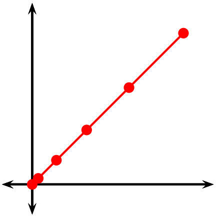

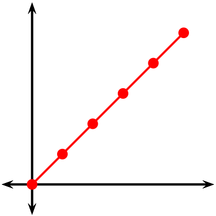

Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

Question

Are the above curves different?

Consider the two parametric curves:

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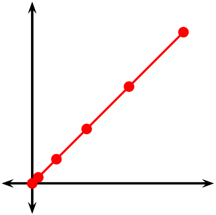

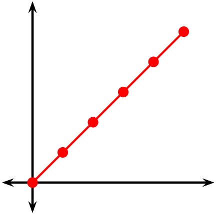

Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

Question

Are the above curves different?

To answer this question we need a definition.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{rcl} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

Definition

A *curve image* (or simply a curve) is any set of points that arises by traversing some continuous curve. In other words, a curve image is any set that can be written in the form

$$\{(f_1(t), \dots, f_n(t)) \mid t \in [a, b]\} \quad ,$$

for some continuous functions $f_1, \dots, f_n$.

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for some continuous functions $f_1, \dots, f_n$.

If we don't require that the functions be continuous, every set of points will be a curve and the definition would be pointless.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{ccc} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

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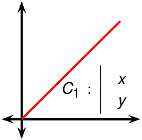
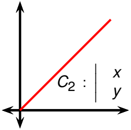
for some continuous functions $f_1, \ldots, f_n$.

Informally, a curve image "remembers" only the points lying on the curve but forgets the "speed" with which each point was visited and "how many times" each point was visited.



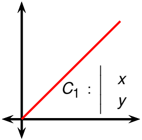
$$C_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1]$$



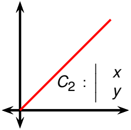
$$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

Are the above curves different?



$C_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right., t \in [0, 1]$



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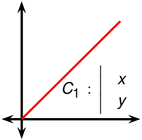
Question

Are the above curves different?

Are the above parametric curves different? Yes.

- As parametric curves, C_1 and C_2 are different: C_1, C_2 are given by different functions.



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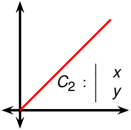


$$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right., t \in [0, 1]$$

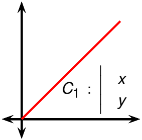
Question

Are the above curves different?

Are the above parametric curves different? Yes.

Are the above curve images different? No.

- As parametric curves, C_1 and C_2 are different: C_1, C_2 are given by different functions.
- As curve images, C_1, C_2 coincide.



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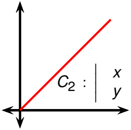


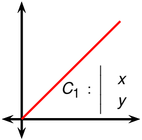
$C_2 : \left| \begin{array}{lll} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$

Question

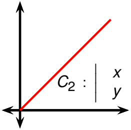
~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- As parametric curves, C_1 and C_2 are different: C_1, C_2 are given by different functions.
- As curve images, C_1, C_2 coincide.
- The original question is incorrectly posed: the word “curve” does not have a mathematical definition without the words “parametric” or “image” attached to it.



$$C_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1]$$



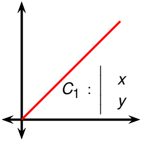
$$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- Nonetheless we sometimes use the word "curve" informally, without specifying "parametric curve" or "curve image".



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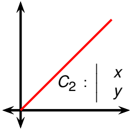


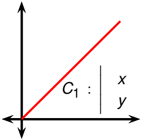
$$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

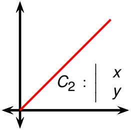
~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- Nonetheless we sometimes use the word “curve” informally, without specifying “parametric curve” or “curve image”.
- In this case, whether we mean “parametric curve” or “curve image” should be clear from the context.



$C_1 : \left| \begin{array}{rcl} x & = & t^2 \\ y & = & t^2 \end{array} \right., t \in [0, 1]$



$C_2 : \left| \begin{array}{rcl} x & = & t \\ y & = & t \end{array} \right., t \in [0, 1]$

Question

~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- Nonetheless we sometimes use the word “curve” informally, without specifying “parametric curve” or “curve image”.
- In this case, whether we mean “parametric curve” or “curve image” should be clear from the context. If not, we are using mathematical language incorrectly.

Graphs of functions as curve images

- Consider a graph of a function given by

$$y = f(x)$$

Graphs of functions as curve images

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- Write $x = t$. Then $y = f(x)$

Graphs of functions as curve images

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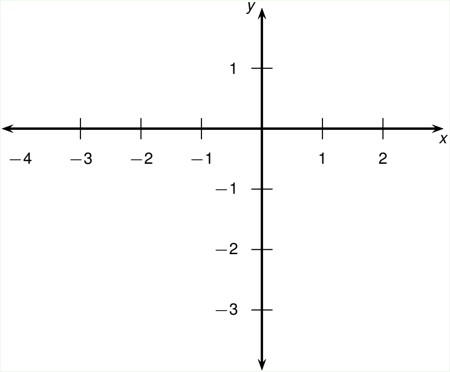
Observation

The graph of an arbitrary function can be written as the image of a curve C using the above transformation.

Example

Sketch and identify the curve image defined by the equations

$$\left| \begin{array}{lcl} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right.$$

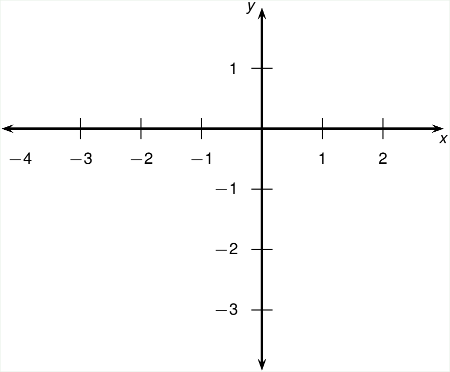


t	x	y
-2		
-1		
0		
1		
2		

Example

Sketch and identify the curve image defined by the equations

$$\left| \begin{array}{lcl} x & = & -t^2 + 2 \\ y & = & t - 1 \end{array} \right.$$



t	x	y
−2		
−1		
0		
1		
2		

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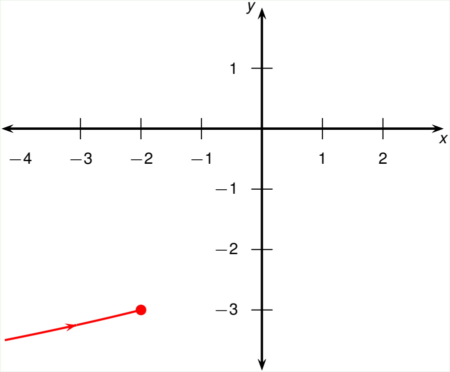

t	x	y
-2	-2	-3
-1		
0		
1		
2		

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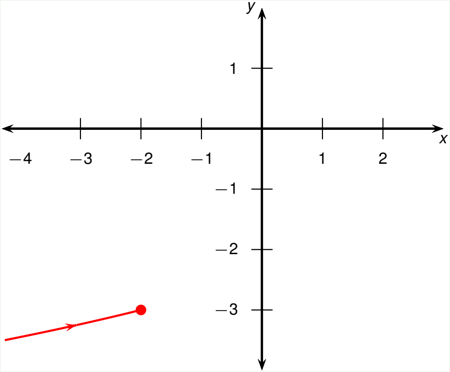

t	x	y
-2	-2	-3
-1		
0		
1		
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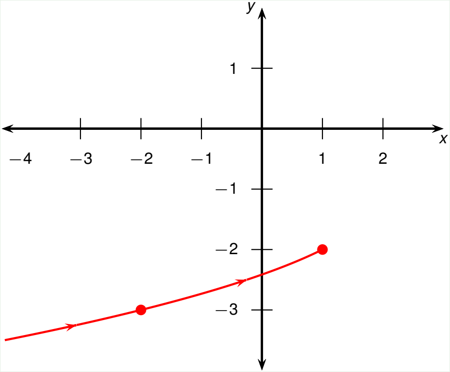

t	x	y
−2	−2	−3
−1	1	−2
0		
1		
2		

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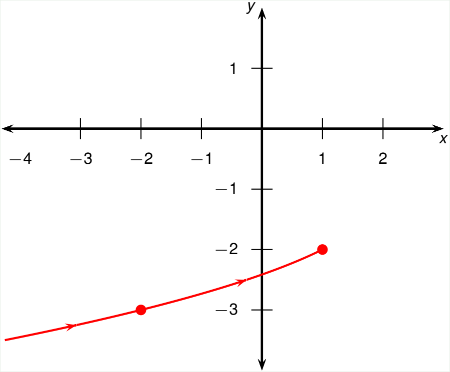
$$\left| \begin{array}{lll} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right.$$



t	x	y
-2	-2	-3
-1	1	-2
0		
1		
2		

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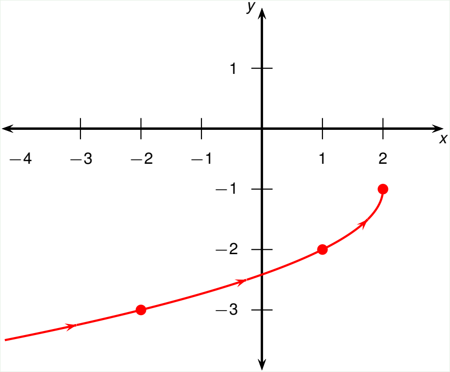

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0	2	-1
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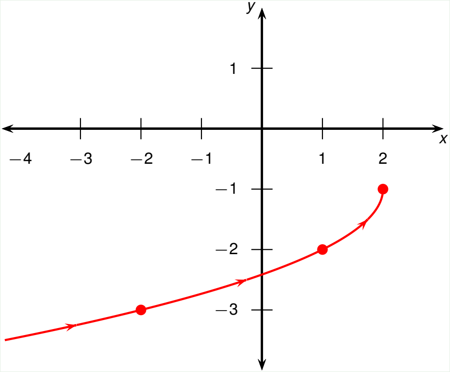

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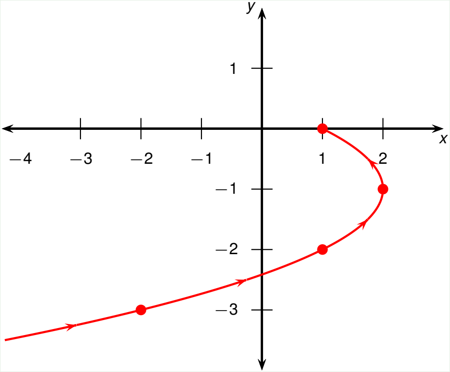

t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
2		

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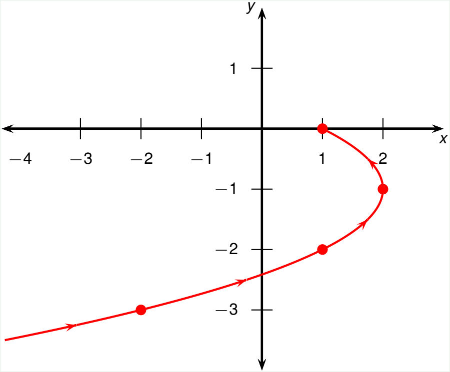

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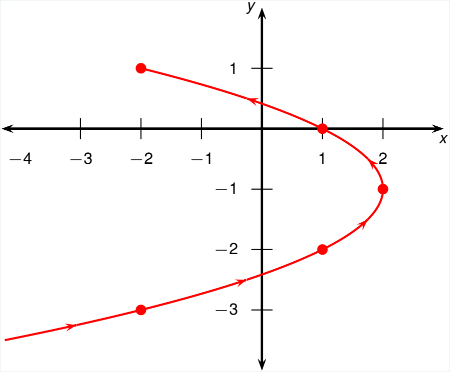

t	x	y
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-1	1	-2
0	2	-1
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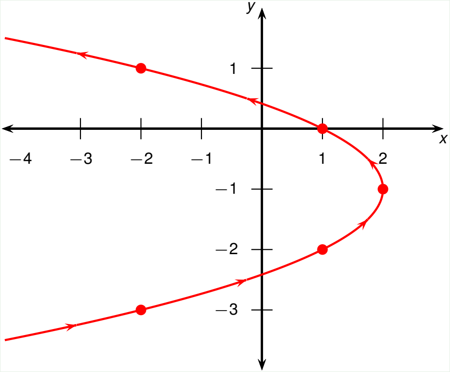

t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
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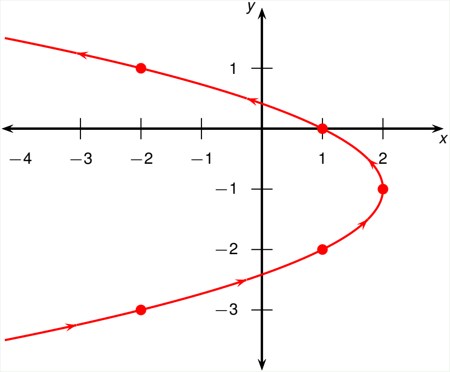
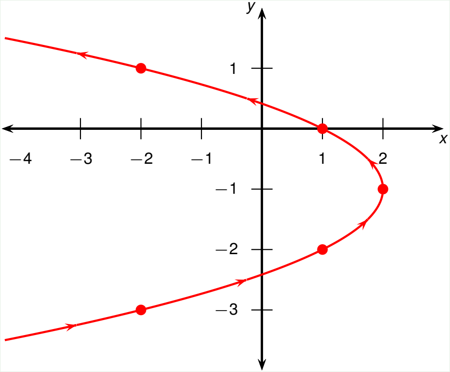


t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
2	-2	1

Example

Sketch and identify the curve image defined by the equations

$$\begin{vmatrix} x & = & -t^2+2 \\ y & = & t-1 \end{vmatrix}$$



t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
2	-2	1

Eliminate t: from second equation we have $t = y + 1$

Example

Sketch and identify the curve image defined by the equations

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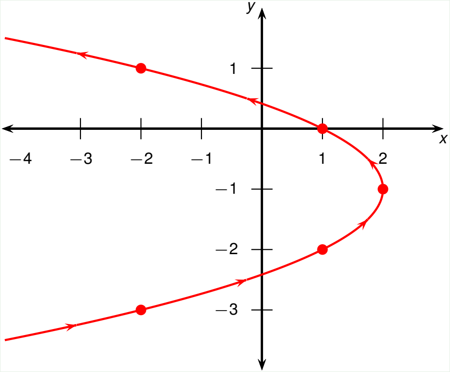

t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
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2	-2	1

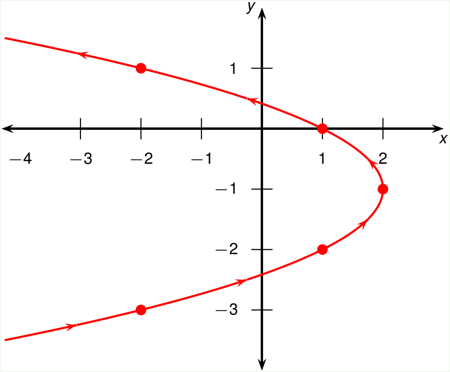
Eliminate t: from second equation we have $t = y + 1$ and therefore:

$$x = -t^2 + 2$$

Example

Sketch and identify the curve image defined by the equations

$$\begin{vmatrix} x & = & -t^2+2 \\ y & = & t-1 \end{vmatrix}$$



t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
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Eliminate t: from second equation we have $t = y + 1$ and therefore:

$$\begin{aligned} x &= -t^2+2 \\ &= -(y+1)^2+2 \end{aligned}$$

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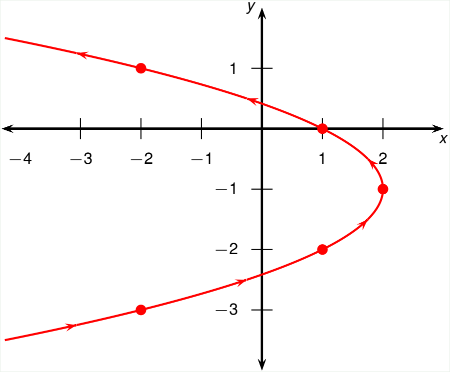
Sketch and identify the curve image defined by the equations

$$\left|\begin{array}{lll} x & = & -t^2+2 \\ y & = & t-1 \end{array}\right.$$



t	x	y
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1	1	0
2	-2	1

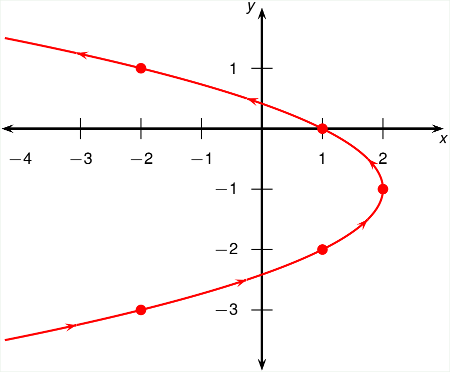
Eliminate t: from second equation we have $t = y + 1$ and therefore:

$$\begin{array}{lll} x & = & -t^2+2 \\ & = & -(y+1)^2+2 \\ & = & -y^2-2y+1 \end{array}$$

Example

Sketch and identify the curve image defined by the equations

$$\begin{cases} x = -t^2 + 2 \\ y = t - 1 \end{cases}$$



t	x	y
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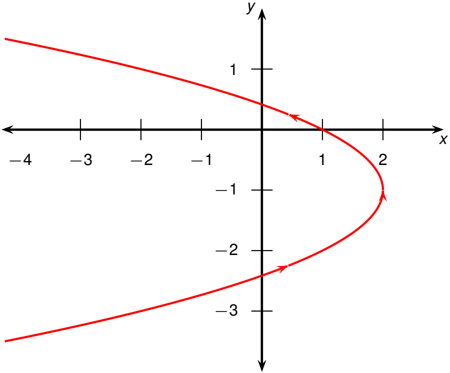
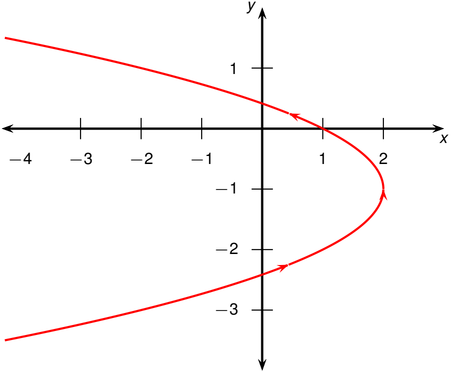
Thus our curve image is a parabola, as expected.



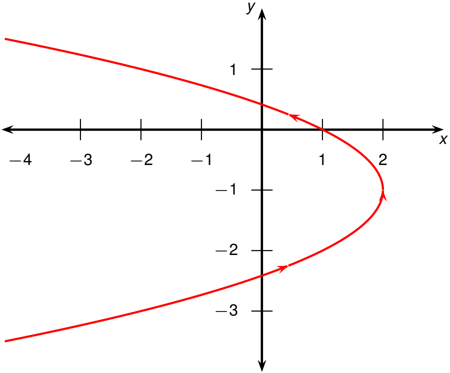
$$\begin{vmatrix} x & = & -t^2 + 2 \\ y & = & t - 1 \end{vmatrix}$$

- There was no restriction placed on t in the last example.



$$\begin{vmatrix} x & = & -t^2+2 \\ y & = & t-1 \end{vmatrix}$$

- There was no restriction placed on t in the last example.
- In such a case we assume $t \in (-\infty, \infty)$, i.e., t runs over all real numbers.



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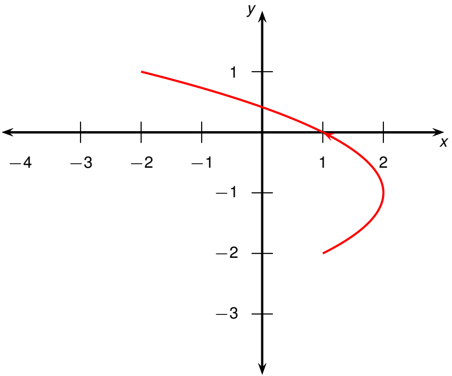
- In general we are expected to specify the interval in which t lies.



$$\left| \begin{array}{rcl} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right., -1 \le t \le 2$$

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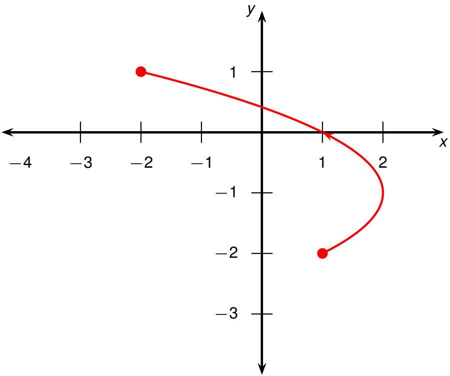
- In general we are expected to specify the interval in which t lies.
- For example, if we restrict the previous example to $t \in [-1, 2]$, we get the part of the parabola that begins at $(1, -2)$ and ends at $(-2, 1)$.



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- For example, if we restrict the previous example to $t \in [-1, 2]$, we get the part of the parabola that begins at $(1, -2)$ and ends at $(-2, 1)$.
- We say that $(1, -2)$ is the initial point and $(-2, 1)$ is the terminal point of the curve.

Implicit vs Explicit (Parametric) Curve Equations

- Consider the parametric curve

$$\left| \begin{array}{rcl} x & = & -t^2+2 \\ y & = & t-1 \quad . \end{array} \right.$$

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$$x + (y + 1)^2 - 2 = 0$$

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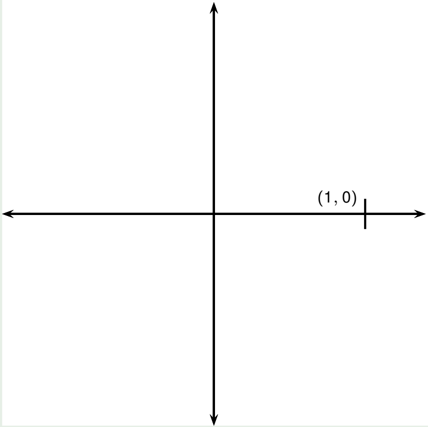
- Equations of the first form are called explicit (parametric) curve equations.
- Equations of the second form are called implicit equations of the curve image.
- Explicit (parametric) curve equations have the advantage that it is easy to generate points on the curve.
- Implicit curve equations have the advantage that it is easy to check whether a point is on the curve.

Example

Sketch and identify the curve defined by the parametric equations

$$x = \cos t, \qquad y = \sin t.$$



t	x	y
0		
$\frac{\pi}{6}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

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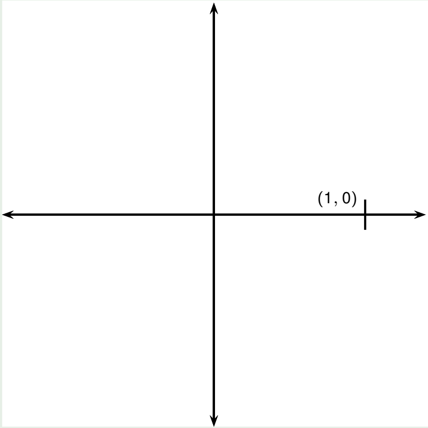

t	x	y
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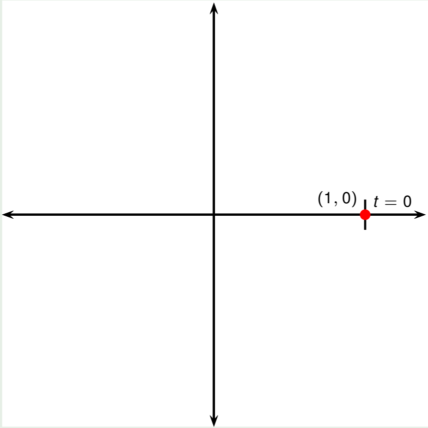

t	x	y
0	1	0
$\frac{\pi}{6}$		
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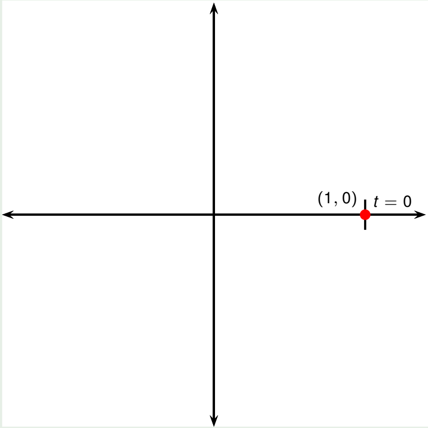

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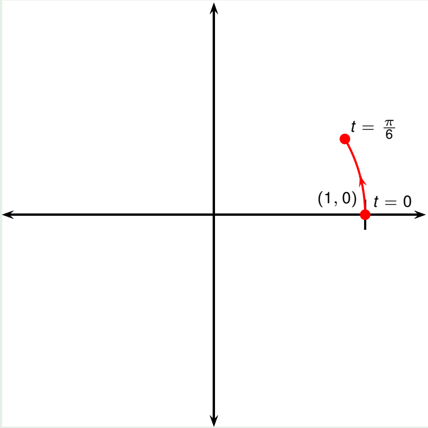

t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		
π		
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2π		

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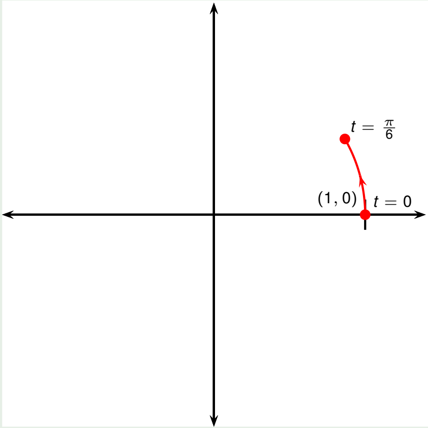

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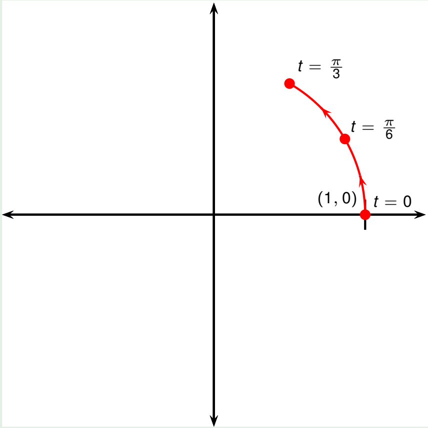

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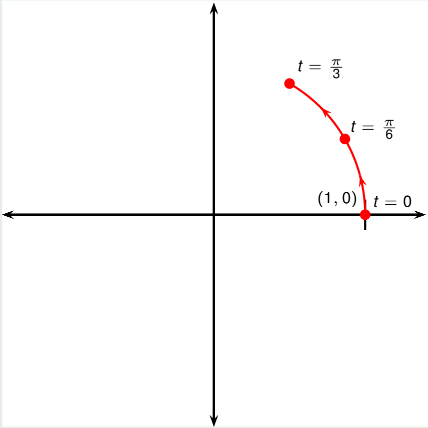
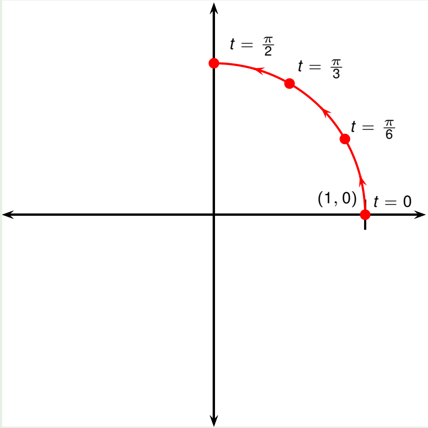


t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

Example

Sketch and identify the curve defined by the parametric equations

$$x = \cos t, \qquad y = \sin t.$$



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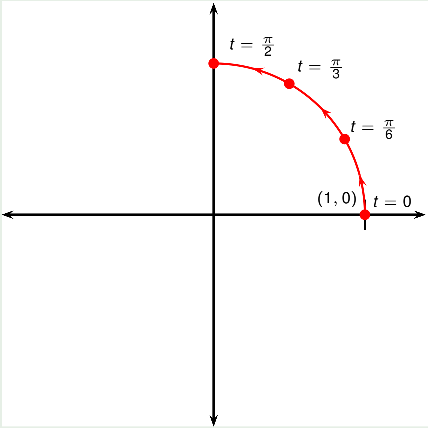

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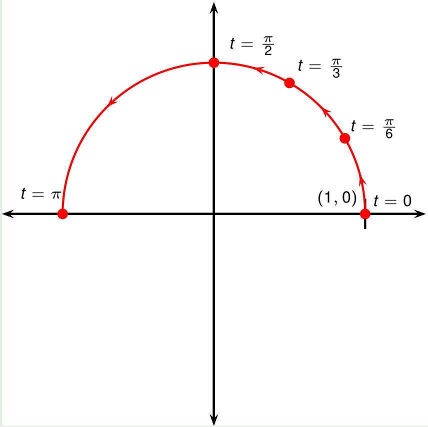

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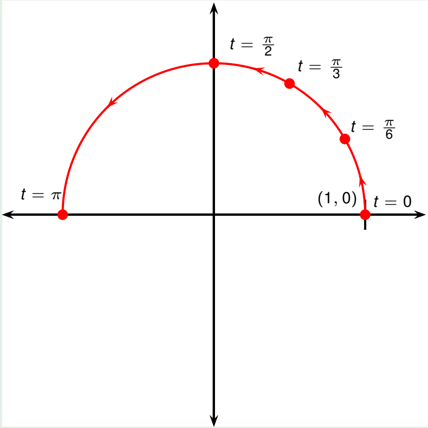

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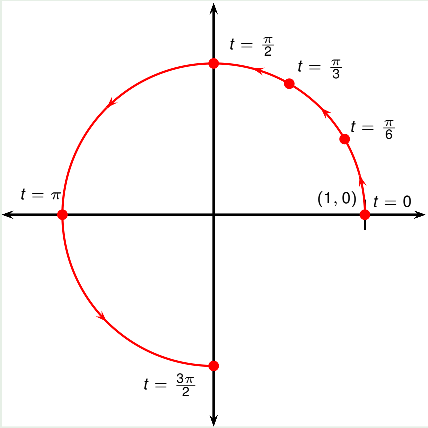

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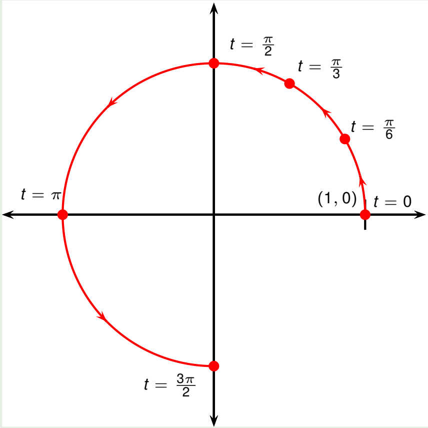

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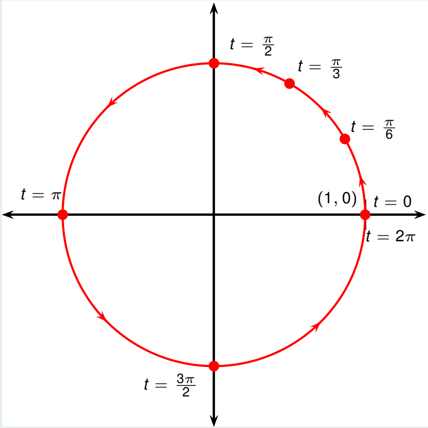

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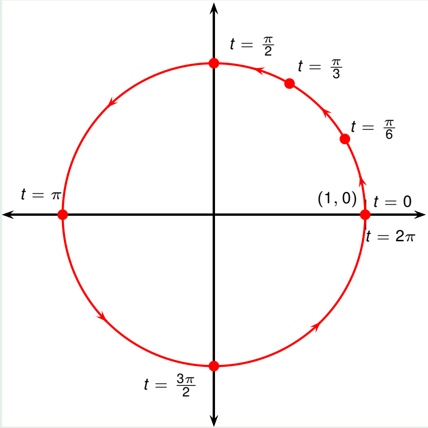

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$$x^2 + y^2 =$$

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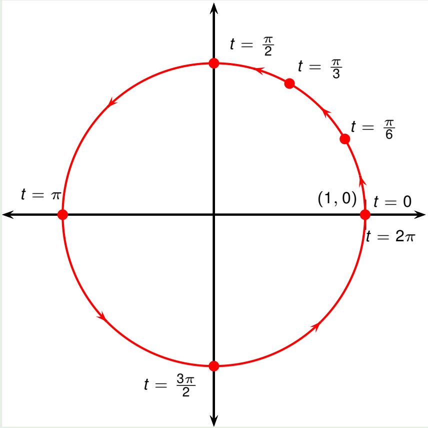

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$$x^2 + y^2 = \cos^2 t + \sin^2 t =$$

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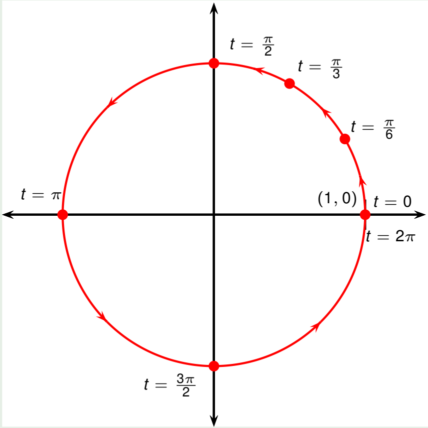

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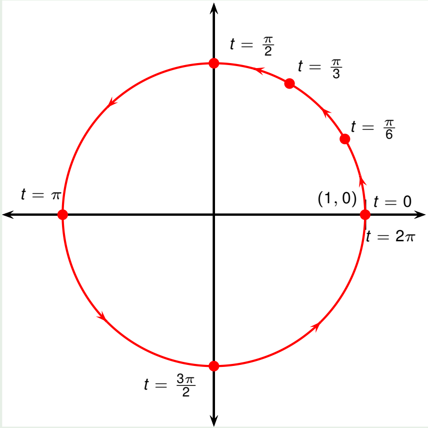

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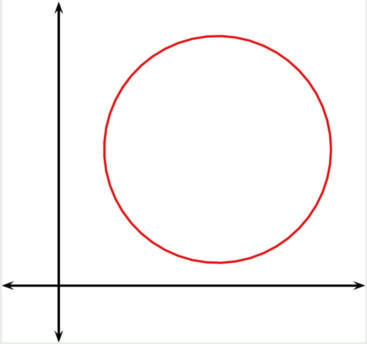
$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Therefore (x, y) travels on the unit circle $x^2 + y^2 = 1$.

Example

Find parametric equations for the circle with center (h, k) and radius r.



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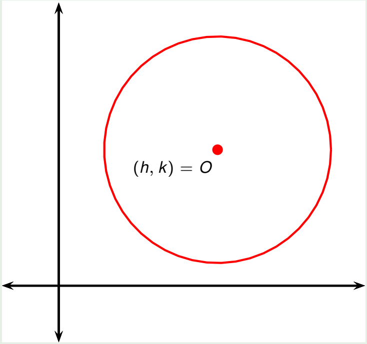

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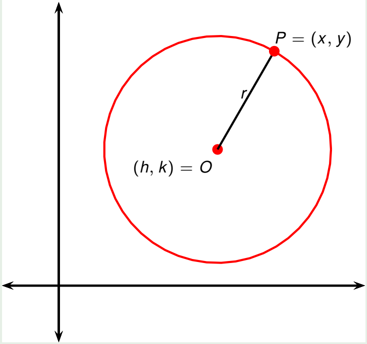

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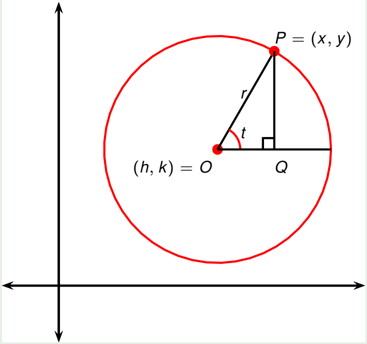

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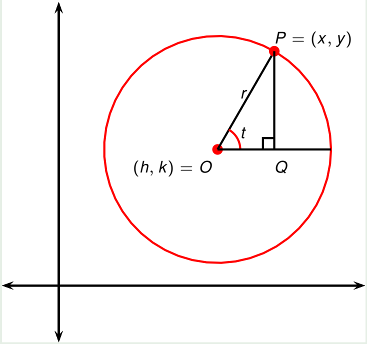

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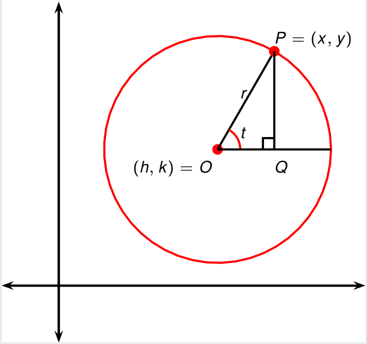

- Let O be the center of the circle with coordinates (h, k).
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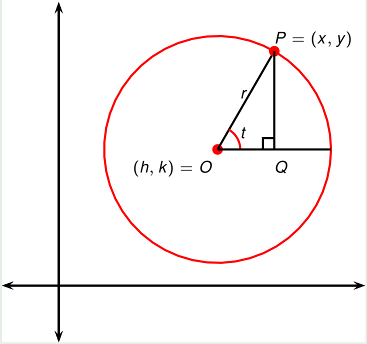

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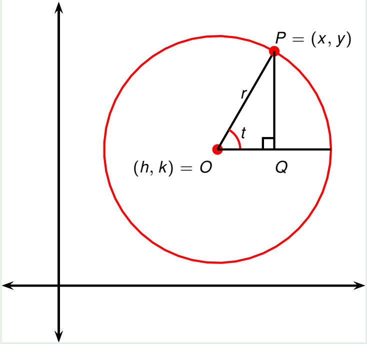

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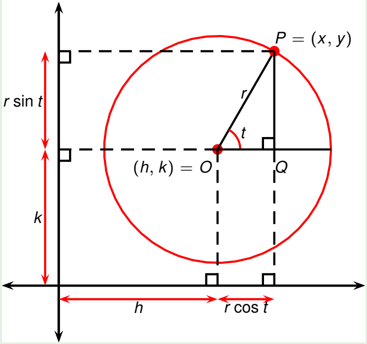

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- Then the coordinates of P are $(h + r\cos t, k + r\sin t)$.

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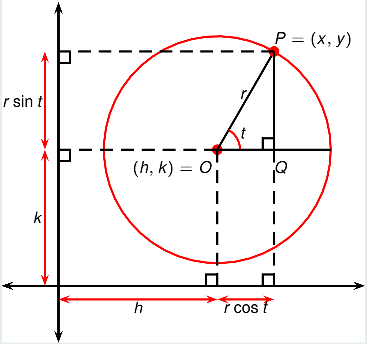

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- Then $|OQ| = \quad r\cos t$.
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- Then the coordinates of P are $(h + r\cos t, k + r\sin t)$.
- In this way we get the parametric equations $\left| \begin{array}{l} x = h + r\cos t \\ y = k + r\sin t \end{array} \right.$, $t \in [0, 2\pi]$

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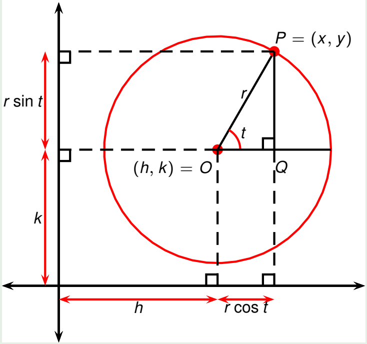

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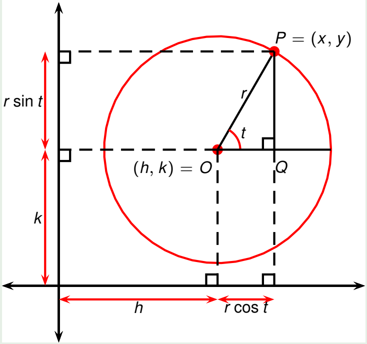

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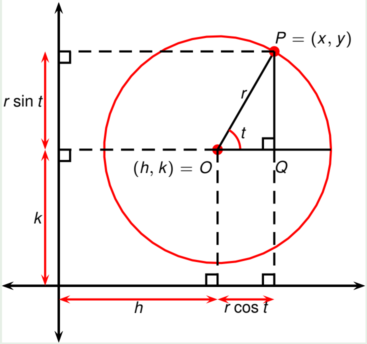

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- Multiply by r to scale the circle to have radius r: $x = r \cos t, \quad y = r \sin t$.
- Add h to x and k to y to translate the circle h units to the left and k units up: $x = h + r \cos t, \quad y = k + r \sin t$

The Cycloid

Definition (Cycloid)

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.

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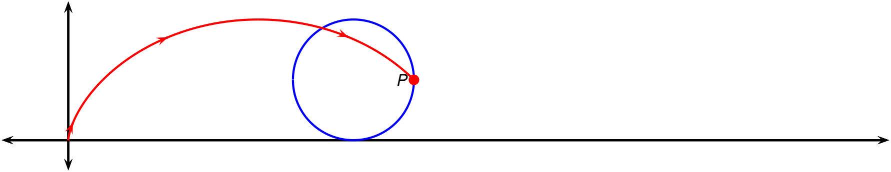

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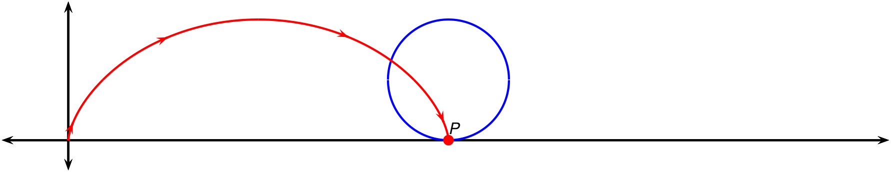

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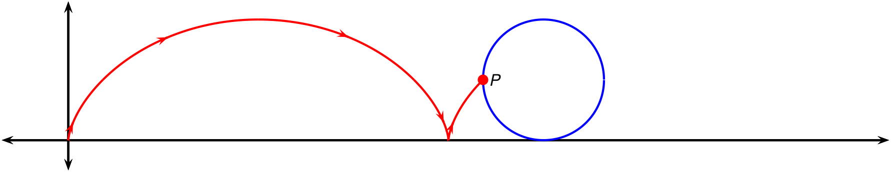

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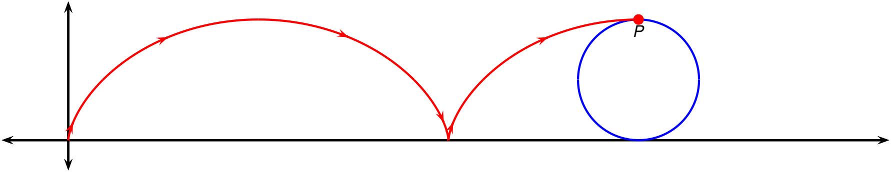

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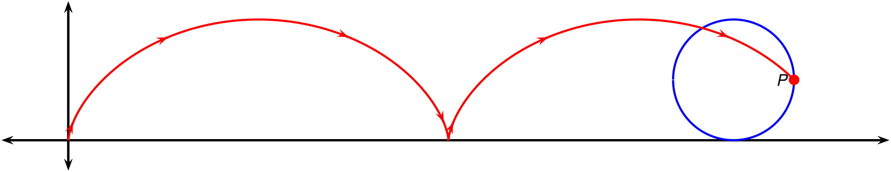

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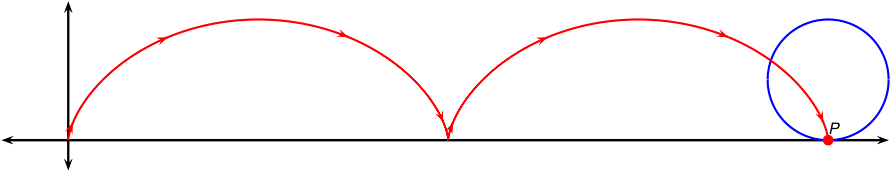

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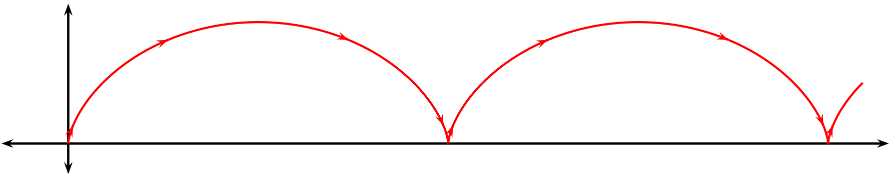

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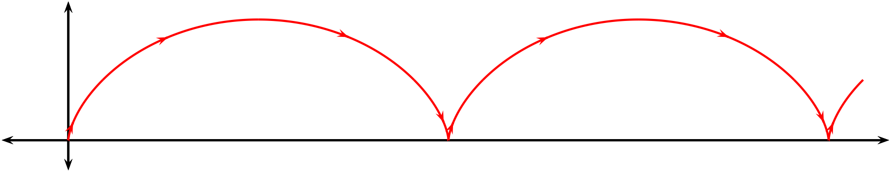

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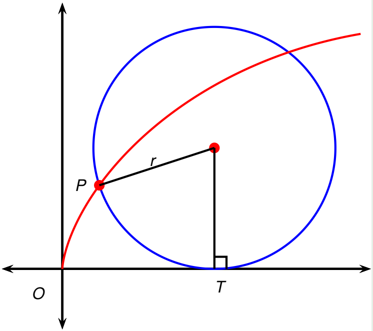
Example

Find parametric equations of a cycloid made using a circle with radius r that rolls along the x-axis such that P hits the origin.



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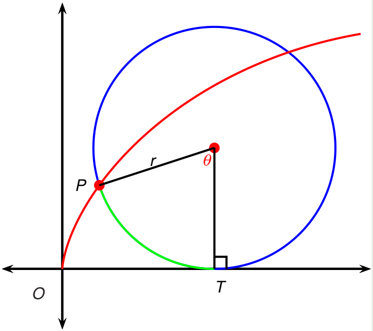

- We choose our parameter to be θ, the angle of rotation of the circle.

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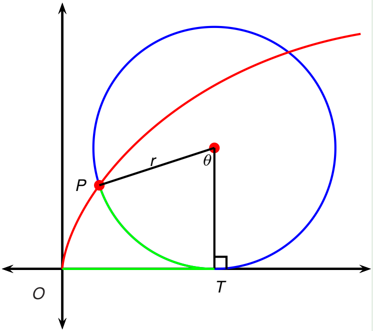

- We choose our parameter to be θ, the angle of rotation of the circle.
- How far has the circle moved if it has rolled through θ radians?

$$|OT| = \operatorname{arc} PT$$

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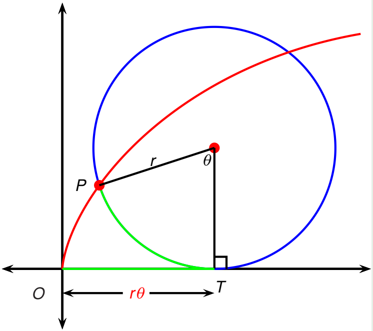

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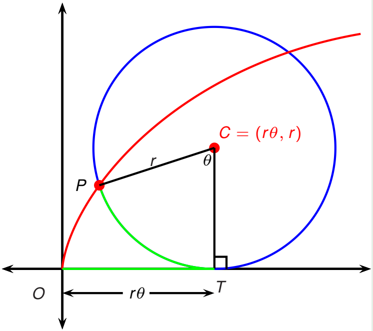

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- Then the center is $C = (r\theta, r)$.

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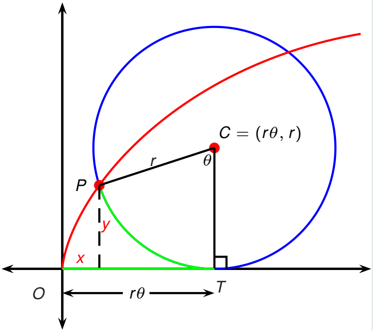

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$$x =$$

$$y =$$

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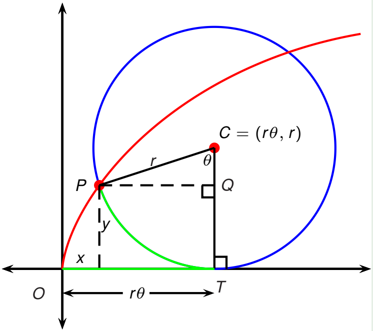

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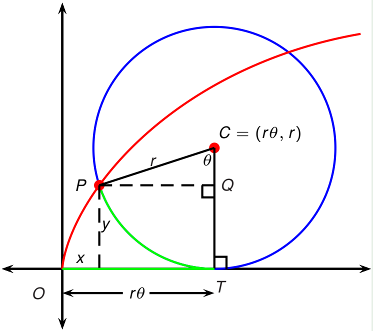

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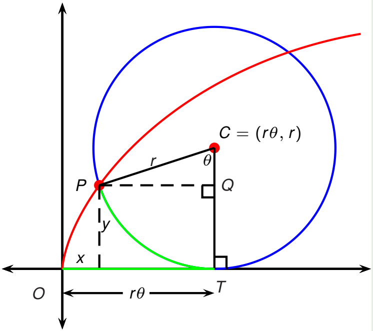

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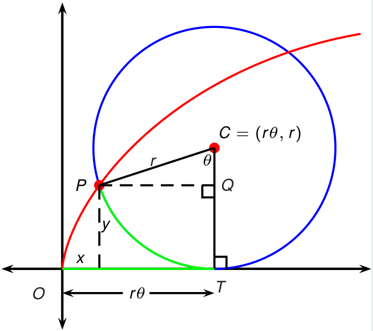

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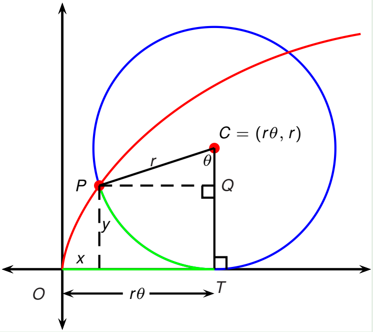

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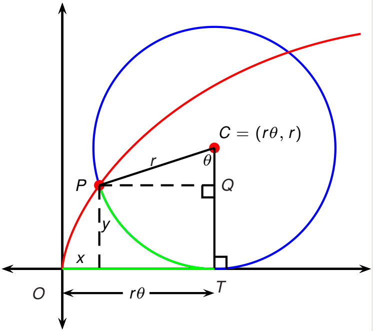

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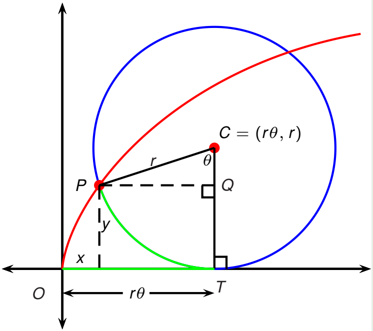

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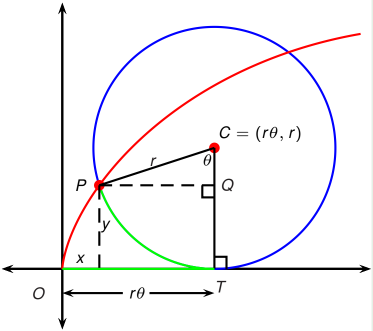

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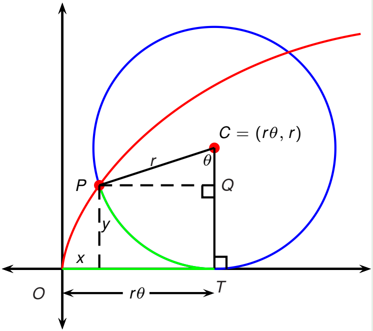

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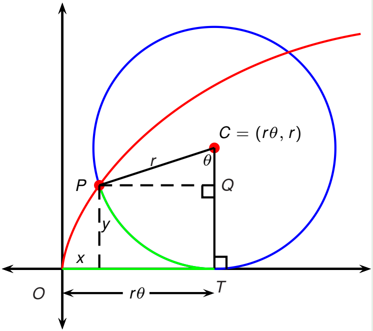

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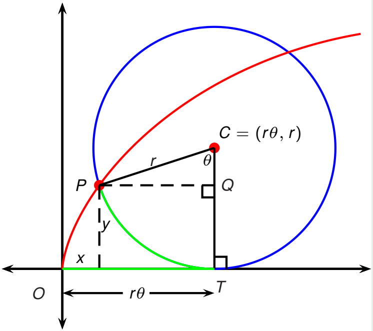

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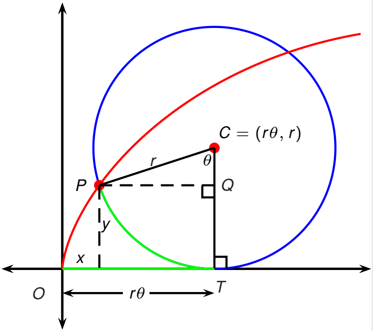

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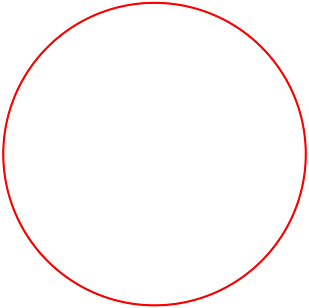
Therefore the equations are

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta), \qquad \theta \in \mathbb{R}$$

Arc Length



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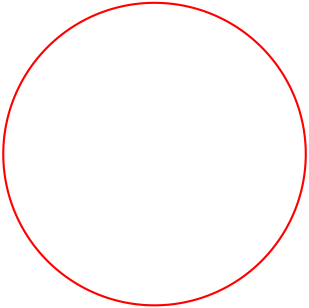

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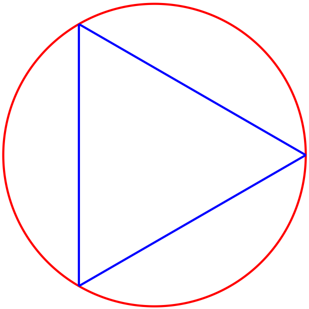

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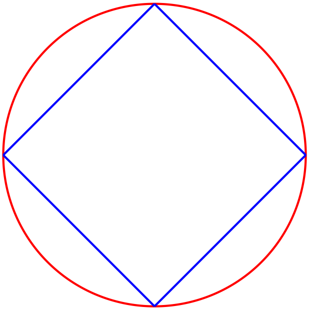

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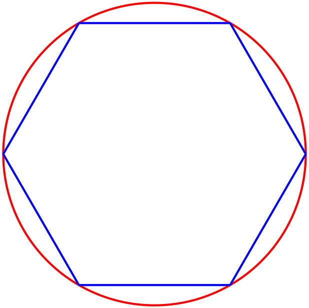

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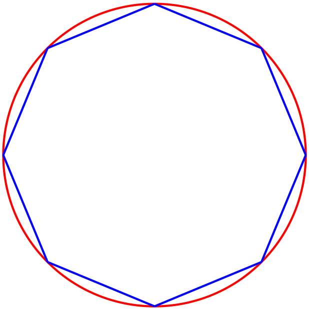

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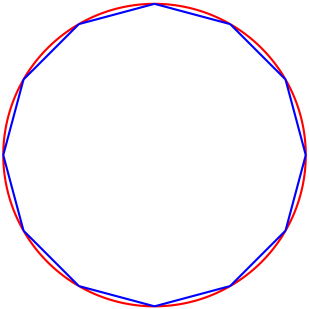

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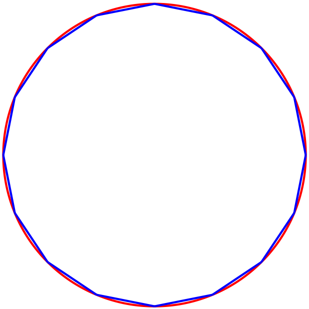

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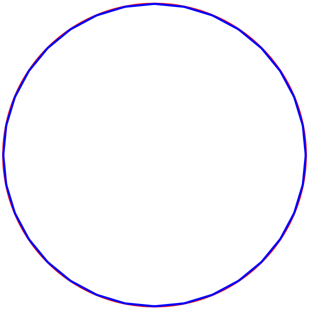

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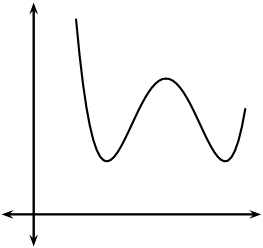

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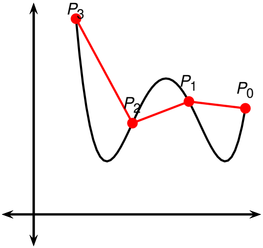

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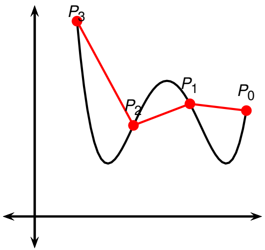

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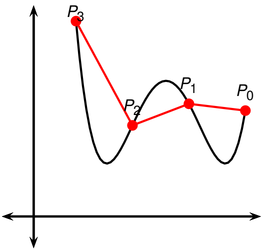

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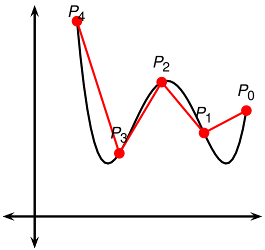

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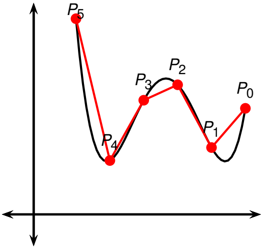

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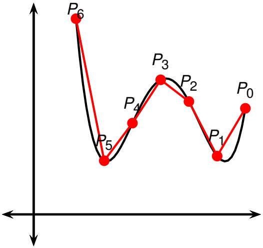

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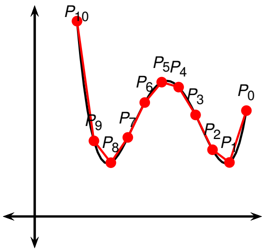

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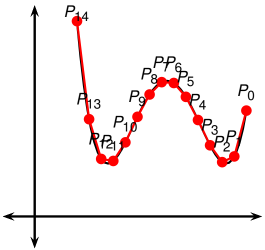

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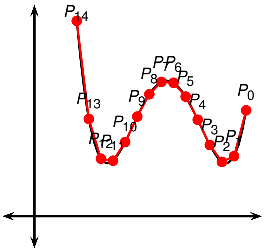

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- Mean Value Theorem: there exist numbers s_i and r_i between t_{i-1} and t_i such that $x(t_i) - x(t_{i-1}) = x'(s_i)(t_i - t_{i-1})$ and $y(t_i) - y(t_{i-1}) = y'(r_i)(t_i - t_{i-1})$

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The Arc Length Formula

Let $\gamma : \left| \begin{array}{lcl} x & = & x(t) \\ y & = & y(t) \end{array} \right. , t \in [a, b]$.

Definition

Suppose $x'(t)$ and $y'(t)$ (exist and) are continuous on $[a, b]$. Then the length of the curve γ is defined as

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

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Arc length of graph of a function

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The Arc Length Formula

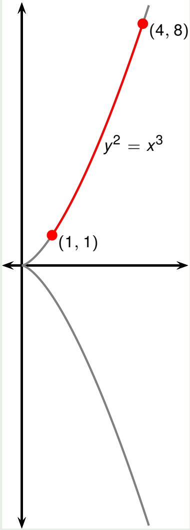
Definition

Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$
$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{(in Leibniz notation)} \quad .$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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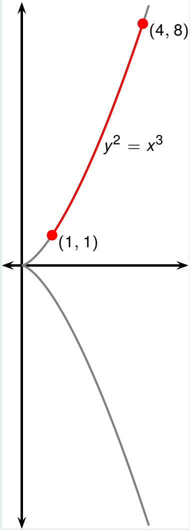

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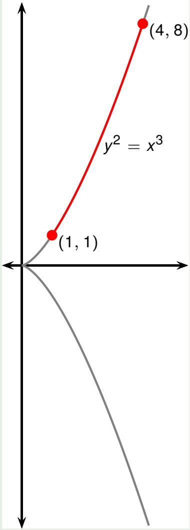

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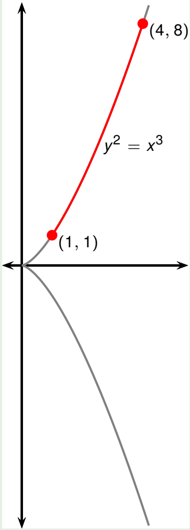

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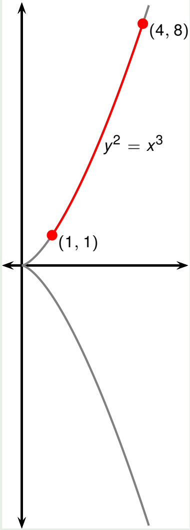

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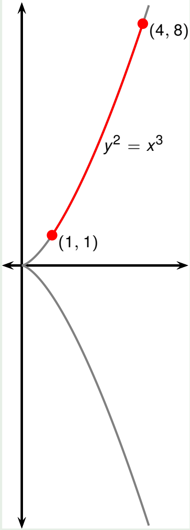

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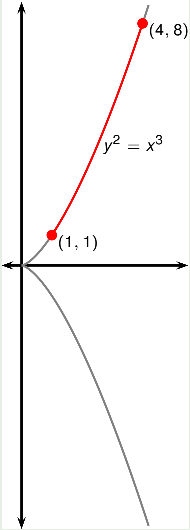

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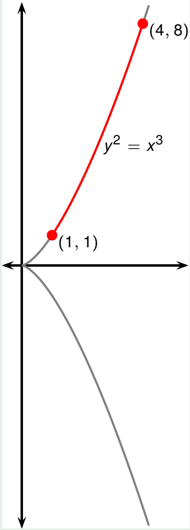

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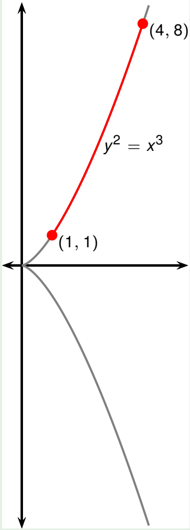

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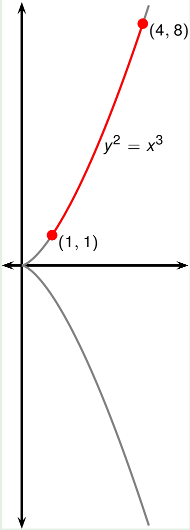

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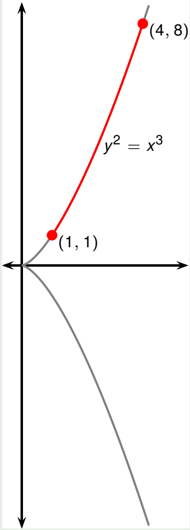
$$= \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \int$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.



- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $du =$.
- When $x = 1$, $u =$.
- When $x = 4$, $u =$.

$$L = \int_1^4 \sqrt{1 + (y')^2}dx$$

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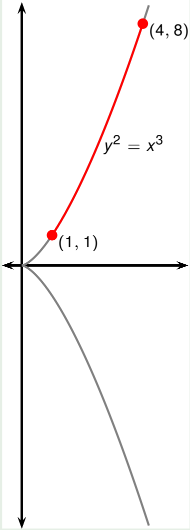

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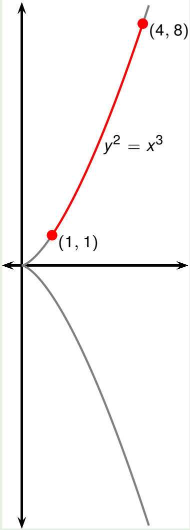

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$$L = \int_1^4 \sqrt{1 + (y')^2}dx$$

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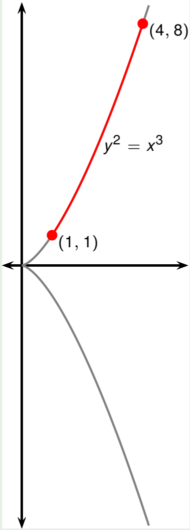

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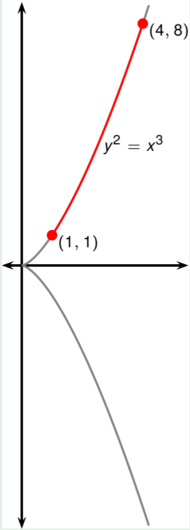
Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



- For the top half of the curve we have:
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- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = \quad$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4} \frac{4}{9}\sqrt{u}\,\mathrm{d}u
\end{aligned}
$$

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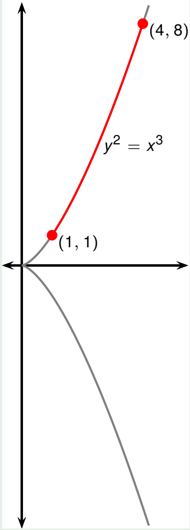

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$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^{4} \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \int_{13/4} \frac{4}{9}\sqrt{u}\, \mathrm{d}u
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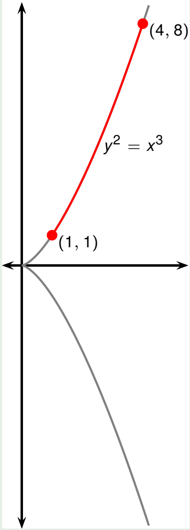
Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



- For the top half of the curve we have:
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- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u
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$$

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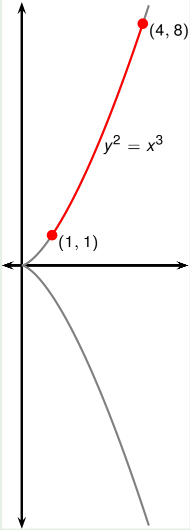
Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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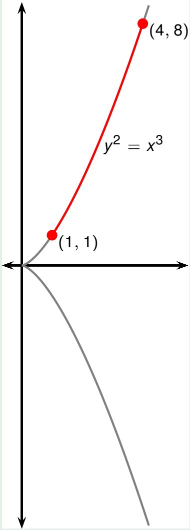
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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$$\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\
&= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)
\end{aligned}$$

If a curve has equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then we can get the length of the curve by interchanging the roles of x and y in the arc length formula:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = \quad$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + \quad}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $dx/dy = \quad$.

$$L = \int_0^1 \sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 \sqrt{1 + \quad}\, dy$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $dx/dy = 2y$.

$$L = \int_0^1 \sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y =$, so $\mathrm{d}y =$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y =$, so $\mathrm{d}y =$, and $\sqrt{1 + 4y^2} =$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + 4y^2}\,\mathrm{d}y$$

$$= \int$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y =$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
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- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} =$.

$$
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L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int \qquad \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta
\end{aligned}
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Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $dx/dy = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $dy = \frac{1}{2}\sec^2\theta d\theta$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1+(dx/dy)^2}\, dy = \int_0^1 \sqrt{1+4y^2}\, dy \\
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$$
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&= \int \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta
\end{aligned}
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- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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- When $y = 0$, $\tan\theta =$, so $\theta =$.
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- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta =$, so $\theta =$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y$$

$$= \int_0 \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta\mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
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$$
\begin{aligned}
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- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \arctan(2)$ (call this α).

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + 4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $dx/dy = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $dy = \frac{1}{2}\sec^2\theta d\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
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$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 \sqrt{1+4y^2}\, dy \\ &= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta \end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\, \mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\, \mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\, \mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, \mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta =$ arctan(2) (call this α).

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1+(\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(\quad + \ln|\quad + \quad|\right)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $dx/dy = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $dy = \frac{1}{2}\sec^2\theta d\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \arctan(2)$ (call this α).

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (dx/dy)^2}\, dy = \int_0^1 \sqrt{1+4y^2}\, dy \\ &= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta \\ &= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\ &= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\ &= \frac{1}{4}\left(2 \quad + \ln| \quad + 2|\right) \end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

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- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
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&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(2\qquad + \ln|\qquad + 2|\right)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

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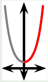
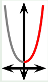
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(2\sqrt{5} + \ln|\sqrt{5} + 2|\right)
\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$y' =$

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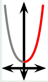

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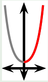

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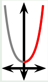

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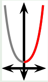

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$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

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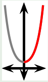

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$$= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.$$

$$L = \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x$$

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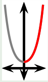

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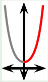
$$= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.$$

$$L = \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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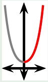

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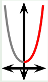
$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

$$\begin{aligned}L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\ &= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\end{aligned}$$

Example $((a+b)^2, (a-b)^2, 2ab = 1/2)$



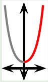
Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

$$\begin{aligned}L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1+\frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\ &= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\ &= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

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$$L = \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x$$

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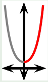
$$= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \Big[\qquad\qquad \Big]_0^1$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$$= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.$$

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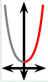

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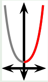

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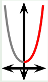

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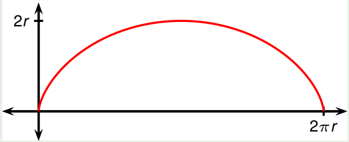
$$\begin{aligned}L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\ &= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\ &= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1 = \frac{e^3 - e^{-3}}{6}.\end{aligned}$$

Example



Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

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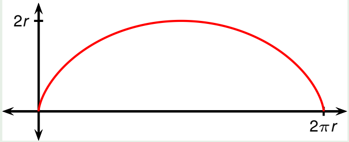

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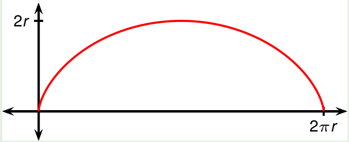

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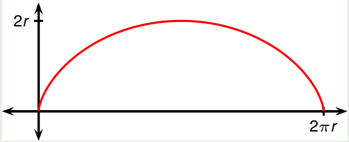

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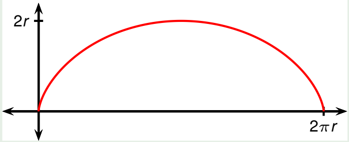

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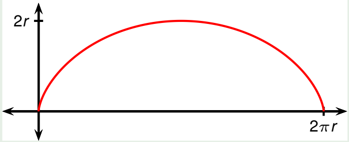

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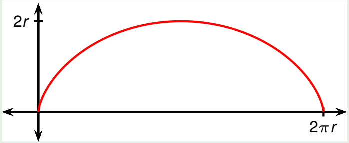

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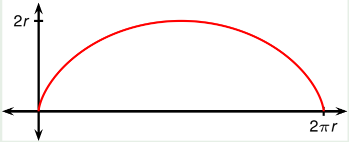

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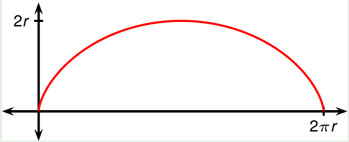

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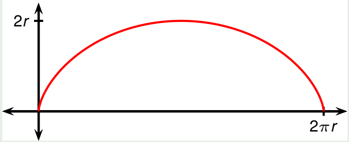

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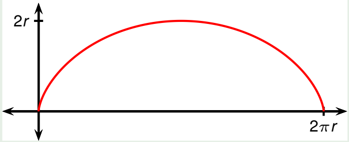

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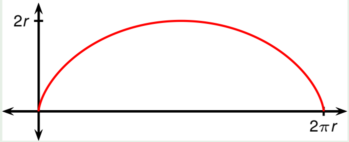

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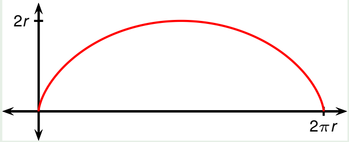

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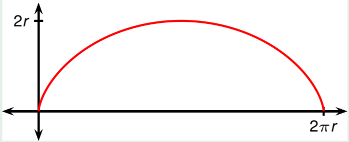

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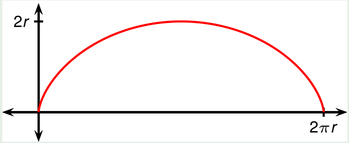

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