Math 141

Lecture 16[material reduced]

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Outline

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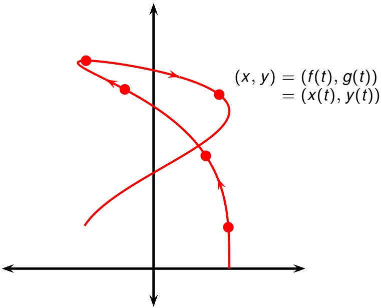
Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.
- The x-coordinate and y-coordinate of the particle are some functions of the time t.
- We can write $x = f(t)$ and $y = g(t)$.
- Less formally, we may directly write $(x, y) = (x(t), y(t))$.
- We say that the equations $\left|\begin{array}{rcl} x & = & f(t) \\ y & = & g(t) \end{array}\right.$ are parametric equations of a parametric curve.
- Note that the curve can't be written as $y = f(x)$: it fails the vertical line test.

Definition (Curve in n-dimensional space)

We define an arbitrary n-tuple of functions $f_1, \ldots, f_n$ on $[a, b]$ to be a *parametric curve* (or simply *curve*). If C is a curve, we write C as:

$$C : \left| \begin{array}{ccc} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

where $x_1, \ldots, x_n$ are the labels of the n-dimensional coordinate system.

Curves in 2- and 3-dimensional space will be of special interest:

A curve in dimension 2 is given by:

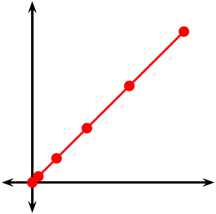
$$C : \left| \begin{array}{ccc} x & = & f(t) \\ y & = & g(t) \end{array} \right. , t \in [a, b] \quad .$$

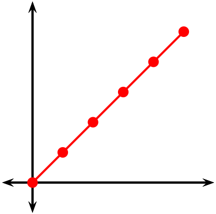
A curve in dimension 3 is given by:

$$C : \left| \begin{array}{ccc} x & = & f(t) \\ y & = & g(t) \\ z & = & h(t) \end{array} \right. , t \in [a, b] \quad .$$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{ccc} x & = & t^2 \\ y & = & t^2 \end{array} \right., t \in [0,1] \qquad \gamma_2 : \left| \begin{array}{ccc} x & = & t \\ y & = & t \end{array} \right., t \in [0,1]$$





Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

Question

Are the above curves different?

To answer this question we need a definition.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{rcl} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

Definition

A *curve image* (or simply a curve) is any set of points that arises by traversing some continuous curve. In other words, a curve image is any set that can be written in the form

$$\{(f_1(t), \dots, f_n(t)) \mid t \in [a, b]\} \quad ,$$

for some continuous functions $f_1, \dots, f_n$.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{ccc} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

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for some continuous functions $f_1, \dots, f_n$.

If we don't require that the functions be continuous, every set of points will be a curve and the definition would be pointless.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{lcl} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

Definition

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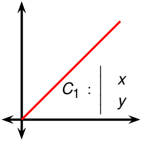
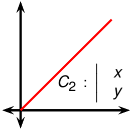
$$\{(f_1(t), \dots, f_n(t)) \mid t \in [a, b]\} \quad ,$$

for some continuous functions $f_1, \dots, f_n$.

Informally, a curve image “remembers” only the points lying on the curve but forgets the “speed” with which each point was visited and “how many times” each point was visited.



$C_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right., t \in [0, 1]$



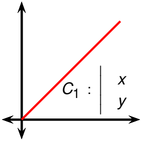
$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right., t \in [0, 1]$

Question

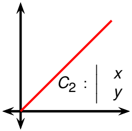
~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- As parametric curves, C_1 and C_2 are different: C_1, C_2 are given by different functions.
- As curve images, C_1, C_2 coincide.
- The original question is incorrectly posed: the word “curve” does not have a mathematical definition without the words “parametric” or “image” attached to it.



$$C_1 : \left| \begin{array}{lcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1]$$



$$C_2 : \left| \begin{array}{lcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

~~*Are the above curves different?*~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- Nonetheless we sometimes use the word “curve” informally, without specifying “parametric curve” or “curve image”.
- In this case, whether we mean “parametric curve” or “curve image” should be clear from the context. If not, we are using mathematical language incorrectly.

Graphs of functions as curve images

- Consider a graph of a function given by

$$y = f(x)$$

- Write $x = t$. Then $y = f(x) = f(t)$, so we get the system

$$C : \left| \begin{array}{rcl} x & = & t \\ y & = & f(t) \end{array} \right. , t \in [a, b]$$

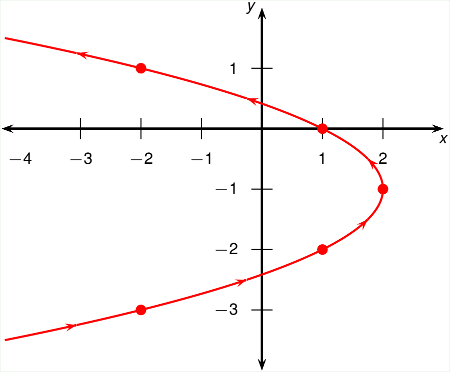
Observation

The graph of an arbitrary function can be written as the image of a curve C using the above transformation.

Example

Sketch and identify the curve image defined by the equations

$$\left| \begin{array}{lll} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right.$$

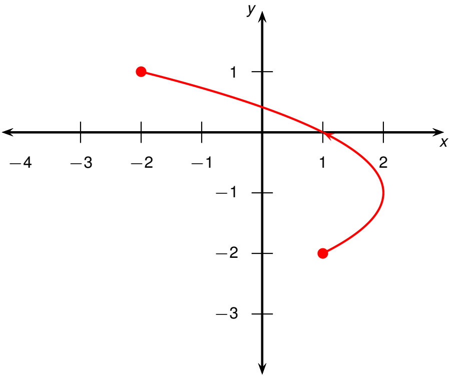


t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
2	-2	1

Eliminate t: from second equation we have $t = y + 1$ and therefore:

$$\begin{array}{lll} x & = & -t^2+2 \\ & = & -(y+1)^2+2 \\ & = & -y^2-2y+1 \end{array}$$

Thus our curve image is a parabola, as expected.



$$\left| \begin{array}{lcl} x & = & -t^2 + 2 \\ y & = & t - 1 \end{array} \right. , -1 \le t \le 2$$

- There was no restriction placed on t in the last example.
- In such a case we assume $t \in (-\infty, \infty)$, i.e., t runs over all real numbers.
- In general we are expected to specify the interval in which t lies.
- For example, if we restrict the previous example to $t \in [-1, 2]$, we get the part of the parabola that begins at $(1, -2)$ and ends at $(-2, 1)$.
- We say that $(1, -2)$ is the initial point and $(-2, 1)$ is the terminal point of the curve.

Implicit vs Explicit (Parametric) Curve Equations

- Consider the parametric curve

$$\left| \begin{array}{lcl} x & = & -t^2 + 2 \\ y & = & t - 1 \end{array} \right. .$$

- As we saw in preceding slides/lectures, all points (x, y) on the image of this curve satisfy the equation

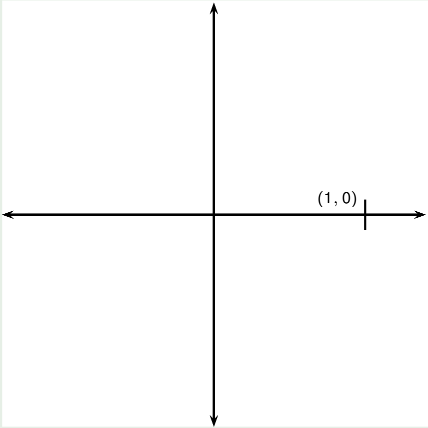
$$x + (y + 1)^2 - 2 = 0$$

- Equations of the first form are called explicit (parametric) curve equations.
- Equations of the second form are called implicit equations of the curve image.
- Explicit (parametric) curve equations have the advantage that it is easy to generate points on the curve.
- Implicit curve equations have the advantage that it is easy to check whether a point is on the curve.

Example

Sketch and identify the curve defined by the parametric equations

$$x = \cos t, \qquad y = \sin t.$$



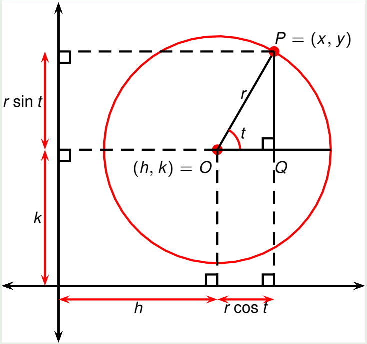
t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Therefore (x, y) travels on the unit circle $x^2 + y^2 = 1$.

Example

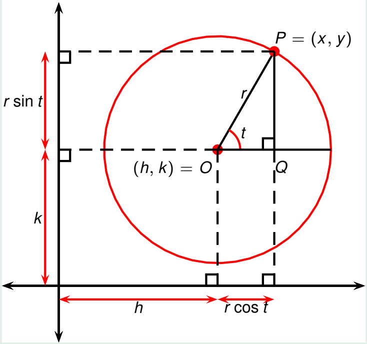
Find parametric equations for the circle with center (h, k) and radius r.



- Let O be the center of the circle with coordinates (h, k).
- Let P be a point on the circle with coordinates (x, y).
- Let t, Q be as indicated on the figure.
- Then $|OQ| = r\cos t$.
- $|PQ| = r\sin t$.
- Then the coordinates of P are $(h + r\cos t, k + r\sin t)$.
- In this way we get the parametric equations $\left|\begin{array}{l} x = h + r\cos t \\ y = k + r\sin t \end{array}\right., t \in [0, 2\pi]$

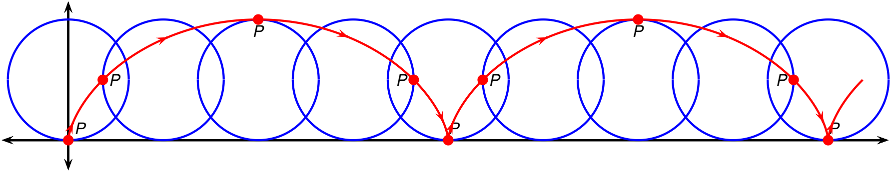
Example

Find parametric equations for the circle with center (h, k) and radius r.



- Alternative solution: $x = \cos t$, $y = \sin t$ are parametric equations of the unit circle.
- Multiply by r to scale the circle to have radius r: $x = r\cos t, \quad y = r\sin t$.
- Add h to x and k to y to translate the circle h units to the left and k units up: $x = h + r\cos t, \quad y = k + r\sin t$

The Cycloid

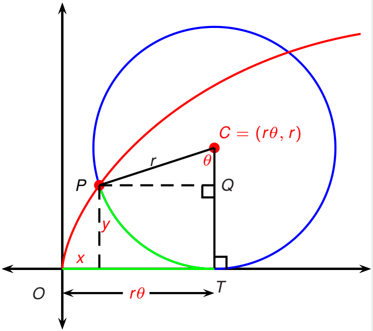


Definition (Cycloid)

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.

Example

Find parametric equations of a cycloid made using a circle with radius r that rolls along the x-axis such that P hits the origin.

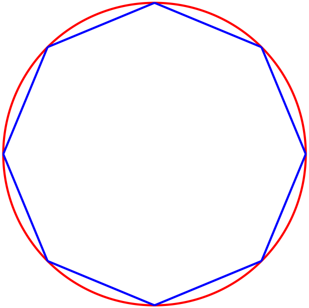


- We choose our parameter to be θ, the angle of rotation of the circle.
- How far has the circle moved if it has rolled through θ radians?
$$|OT| = \text{arc}\, PT = r\theta$$
- Then the center is $C = (r\theta, r)$.
- Let the coordinates of P be (x, y).

$$\begin{aligned} x &= |OT| - |PQ| &= r\theta - r\sin\theta \\ y &= |CT| - |CQ| &= r - r\cos\theta \end{aligned}$$

Therefore the equations are
$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta), \qquad \theta \in \mathbb{R}$$

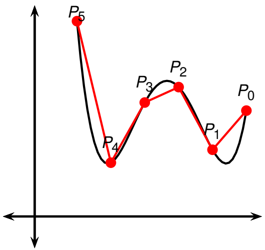
Arc Length



- What do we mean by the length of a curve?
- The length of a polygon is easy to compute: add up the length of the line segments that form the polygon.
- If the curve is a circle, approximate it by a polygon.
- Then take the limit as the number of segments of the polygon goes to ∞.

Let γ be the curve $\gamma : \left| \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right., t \in [a, b]$

- Divide $[a, b]$ into n subintervals with endpoints $t_0, t_1, \ldots, t_n$ and equal width Δt.
- The points $P_i = (x(t_i), y(t_i))$ lie on the curve γ. The lengths of the segments with endpoints with consecutive indices from $P_0, P_1, \ldots, P_n$ approximate the length of the curve γ.
- The length L of the curve γ is the limit of the lengths of these segments as $n \to \infty$.



$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

Let γ be the curve $\gamma : \left| \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right., t \in [a, b]$

$$\begin{aligned} L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\, \Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t \end{aligned}$$

- If f has continuous derivative, we can compute the above limit.
- Let $\left| \begin{array}{l} x_i = x(t_i) \\ y_i = y(t_i) \end{array} \right.$, and $\left| \begin{array}{lcl} \Delta x &=& x_i - x_{i-1} = x(t_i) - x(t_{i-1}) \\ \Delta y &=& y_i - y_{i-1} = y(t_i) - y(t_{i-1}) \end{array} \right.$.
- Then $|P_iP_{i-1}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.
- Mean Value Theorem: there exist numbers s_i and r_i between t_{i-1} and t_i such that $x(t_i) - x(t_{i-1}) = x'(s_i)(t_i - t_{i-1})$ and $y(t_i) - y(t_{i-1}) = y'(r_i)(t_i - t_{i-1})$.
- $\Delta x = x'(s_i)\Delta t$, $\Delta y = y'(r_i)\Delta t$.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x'(s_i)\Delta t)^2 + (y'(r_i)\Delta t)^2} \\ &= \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\sqrt{(\Delta t)^2} = \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\, \Delta t \end{aligned}$$

The Arc Length Formula

Let $\gamma : \left| \begin{array}{rcl} x & = & x(t) \\ y & = & y(t) \end{array} \right. , t \in [a, b]$.

Definition

Suppose $x'(t)$ and $y'(t)$ (exist and) are continuous on $[a, b]$. Then the length of the curve γ is defined as

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t \\ &= \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \qquad \text{in Leibniz notation .} \end{aligned}$$

Arc length of graph of a function

Question

What is the length of the graph of the curve given by the graph of $y = f(x)$?

- The graph of $y = f(x)$ is written as a curve as

$$\gamma : \left| \begin{array}{lcl} x & = & t \\ y & = & f(t) \end{array} \right. , t \in [a, b] \quad .$$

- In other words, the question asks what is the length $L(\gamma)$ of γ. That is a straightforward computation:

$$L(\gamma) \quad = \quad \int \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int \sqrt{1 + (f'(t))^2} \mathrm{d}t$$

The Arc Length Formula

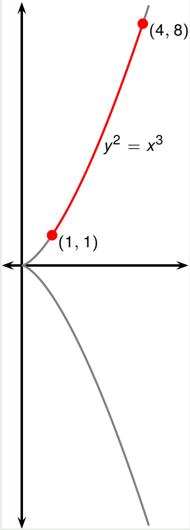
Definition

Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \\ &= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \qquad \text{(in Leibniz notation)} \quad . \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.



- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}u = \frac{9}{4}\mathrm{d}x$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) \end{aligned}$$

If a curve has equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then we can get the length of the curve by interchanging the roles of x and y in the arc length formula:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, \mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\, \mathrm{d}y$$

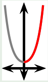
Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \arctan(2)$ (call this α).

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(2\sqrt{5} + \ln|\sqrt{5} + 2|\right)
\end{aligned}
$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$(y')^2 = \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}$$

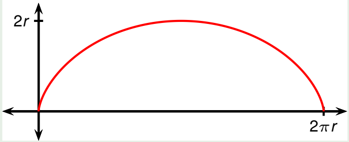
$$= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.$$

$$L = \int_0^1 \sqrt{1 + (y')^2} \mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}} \mathrm{d}x$$

$$= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}} \mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2} \mathrm{d}x$$

$$= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right) \mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1 = \frac{e^3 - e^{-3}}{6}.$$

Example



Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

The first arch is $0 \le \theta \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \mathrm{d}\theta = \int_0^{2\pi} \sqrt{(r(1-\cos\theta))^2 + (r\sin\theta)^2} \mathrm{d}\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} \mathrm{d}\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)} \mathrm{d}\theta$$

Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Then

$$\sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2|\sin(\theta/2)| = 2\sin(\theta/2)$$

$$L = r\int_0^{2\pi} 2\sin(\theta/2)\mathrm{d}\theta = r\left[-4\cos(\theta/2)\right]_0^{2\pi} = 8r$$