Math 141

Lecture 18[material skipped, included on final as bonus]

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Outline

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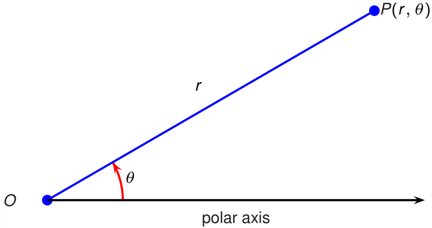
Polar Coordinates

- The polar coordinate system is an alternative to the Cartesian coordinate system.
- Choose a point in the plane called O (the origin).
- Draw a ray starting at O. The ray is called the polar axis. This ray is usually drawn horizontally to the right.



- Let P be a point in the plane.
- Let θ denote the angle between the polar axis and the line OP.
- Let r denote the length of the segment OP.
- Then P is represented by the ordered pair (r, θ).

Polar Coordinates

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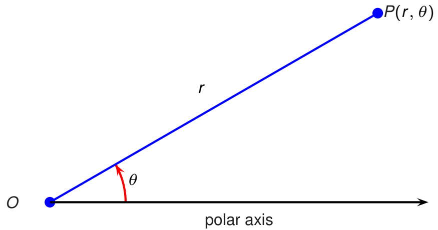
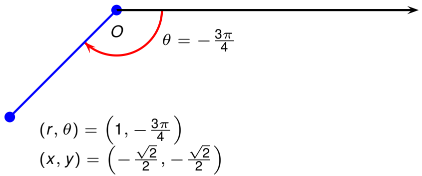
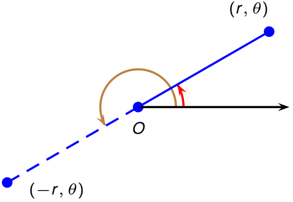


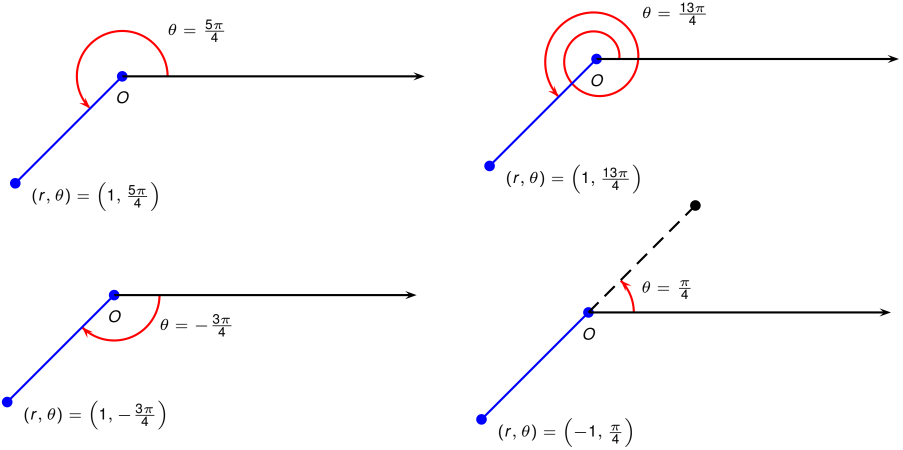
- The letters (x, y) imply Cartesian coordinates and the letters (r, θ)- polar. When we use other letters, it should be clear from context whether we mean Cartesian or polar coordinates. If not, one must request clarification.

1. What if θ is negative?
2. What if r is negative?
3. What if r is 0?





1. Positive angles θ are measured in the counterclockwise direction from O. Negative angles are measured in the clockwise direction.
2. Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O, but on opposite sides.
3. If $r = 0$, then $(0, \theta)$ represents O for all values of θ.



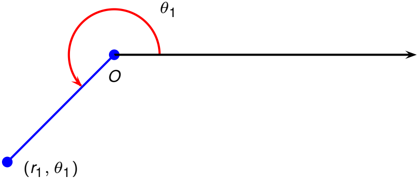
- There are many ways to represent the same point.
- We could use a negative θ.
- We could go around more than once.
- We could use a negative r.

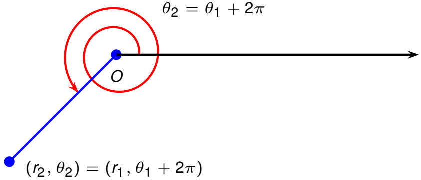
- Let P_1 be point with polar coordinates (r_1, θ_1).
- Let P_2 be point with polar coordinates (r_2, θ_2).

Observation

P_1 coincides with P_2 if one of the three mutually exclusive possibilities holds:

- $r_1 = r_2 \neq 0$ *and* $\theta_2 = \theta_1 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ *and* $\theta_2 = \theta_1 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ *and* θ *is arbitrary.*





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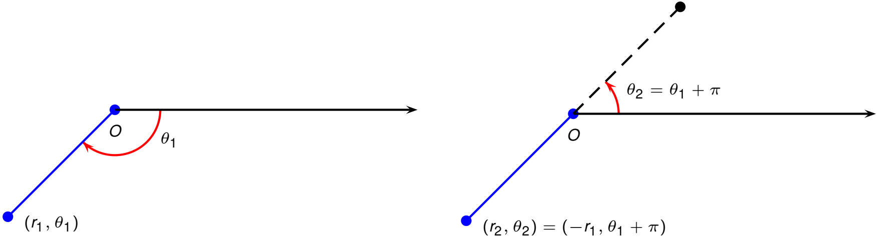
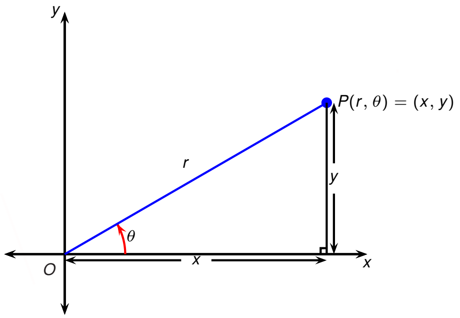


- How do we go from polar coordinates to Cartesian coordinates?
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y).
- How do we go from Cartesian coordinates to polar coordinates?



$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\hline
\cos\theta &= \frac{x}{r} \\
\sin\theta &= \frac{y}{r} \\
r^2 &= x^2 + y^2 \\
\hline
r &= \sqrt{x^2 + y^2} \\
\theta &= \arcsin\left(\frac{y}{r}\right) \quad \text{if } x > 0 \\
&= \arccos\left(\frac{x}{r}\right) \quad \text{if } y > 0 \\
&= \arctan\left(\frac{y}{x}\right) \quad \text{if } x > 0
\end{aligned}
$$

Example

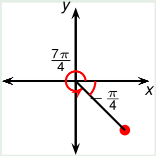
Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Therefore the point with polar coordinates $(2, \frac{\pi}{3})$ has Cartesian coordinates $(1, \sqrt{3})$.

Example



Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- Therefore one possible representation of $(1, -1)$ in polar coordinates is $(\sqrt{2}, 7\pi/4)$.
- $(\sqrt{2}, -\pi/4)$ is another.

$$\begin{aligned} r &= \pm\sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

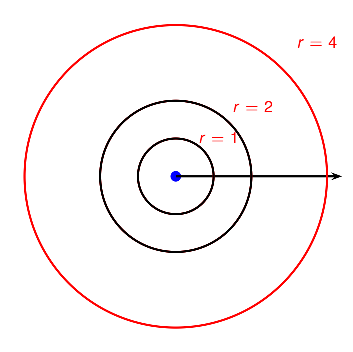
- Recall polar coordinates:

$$\left|\begin{array}{lcl} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array}\right.$$

- A curve in polar coordinates is given by specifying explicit or implicit equations in polar coordinates.

Example

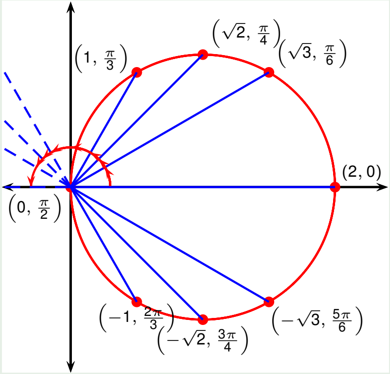
What curve is represented by the polar equation $r = 2$?



- The equation describes all points that are 2 units away from O.
- This is the circle with center O and radius 2.
- The equation $r = 1$ describes the unit circle.
- The equation $r = 4$ describes the circle with center O and radius 4.

Example

1. Sketch the curve with polar equation $r = 2\cos\theta$.
2. Find a Cartesian equation for this curve.



θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

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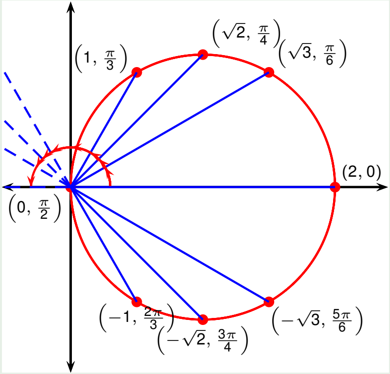
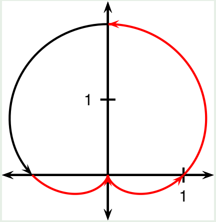


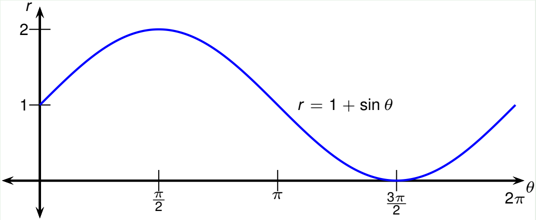
- $x = r\cos\theta$.
- $\cos\theta = x/r$.
- $r = 2\cos\theta = 2x/r$.
- $2x = r^2 = x^2 + y^2$.
- $x^2 + y^2 - 2x = 0$.
- Complete the square:

$$\begin{aligned}(x^2 - 2x + 1) + y^2 &= 0 + 1\\ (x-1)^2 + y^2 &= 1\end{aligned}$$

Example (Cardioid)

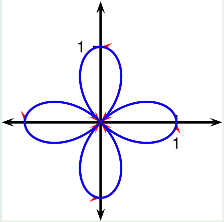
Sketch the curve $r = 1 + \sin\theta$.

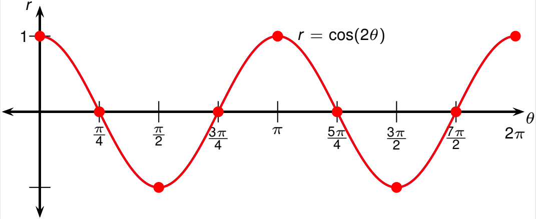




Example

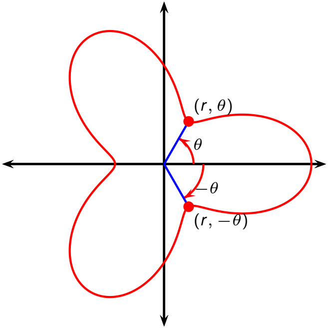
Sketch the curve $r = \cos(2\theta)$.





Symmetry

- If the polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
- If the equation is unchanged when θ is replaced by $\pi + \theta$, the curve is symmetric under rotation about the pole.
- If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$.



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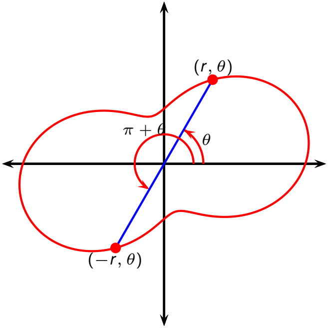

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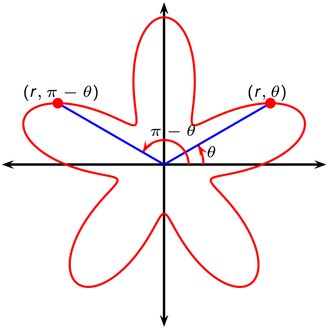
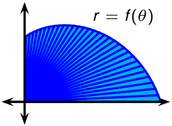


Areas in Polar Coordinates

Suppose we have a polar curve $r = f(\theta)$, $a \leq \theta \leq b$.

Definition

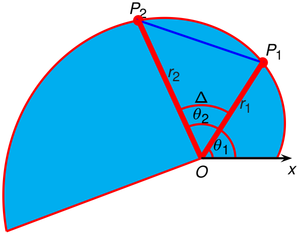
We say that the figure obtained as the union of the segments connecting the origin with the points of the curve is the figure *swept* by the curve as θ varies from a to b.



Theorem

Suppose no two points on the curve lie on the same ray from the origin. Then the area swept by the curve equals $A = \int_a^b \frac{1}{2} (f(\theta))^2 \, d\theta$.

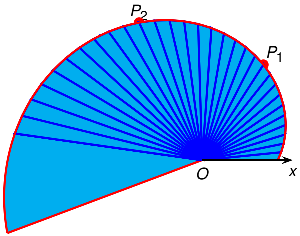
Area swept by a polar curve: justification



Split $[a, b]$ into N equal segments via points $a = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{N-1} \leq \theta_N = b$. The length of each segment is $\Delta = \frac{b-a}{N}$. Let $r_i = f(\theta_i)$. Then each θ_i gives a point P_i with polar coordinates (r_i, θ_i).

The area swept by the curve is approximated by sum of areas of triangles given by connecting the origin with two consecutive vertices. Consider one such triangle, say, OP_1P_2. By Euclidean geometry, the area of $\triangle OP_1P_2$ is $\frac{|OP_1||OP_2| \sin \Delta}{2} = \frac{r_1 r_2 \sin \Delta}{2} = \frac{f(\theta_1) f(\theta_2) \sin \Delta}{2}$.

Area swept by a polar curve: justification



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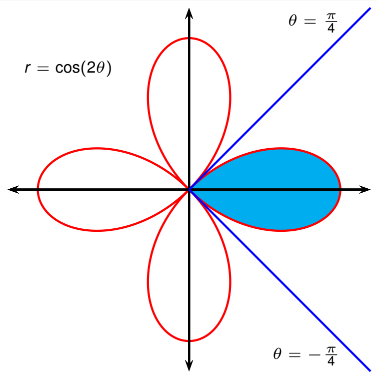
Therefore the area swept by the curve equals the limit of the sum:

$$
\begin{aligned}
A &= \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_{i+1}) \sin \Delta}{2} = \lim_{\Delta \to 0} \frac{\sin \Delta}{\Delta} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i + \Delta) \Delta}{2} \\
\text{(can be proved)} &= \lim_{\Delta \to 0} \frac{\sin \Delta}{\Delta} \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i + \Delta) \Delta}{2} = 1 \cdot \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i) \Delta}{2} \\
\text{(Riemann sum)} &= \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i) \Delta}{2} = \int_a^b \frac{f^2(\theta)}{2} d\theta
\end{aligned}
$$

Example

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

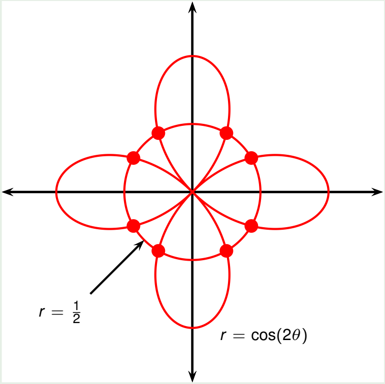


$$
\begin{aligned}
A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \\
&= \int_{0}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \\
&= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) d\theta \\
&= \frac{1}{2} \left[\theta + \frac{1}{4} \sin(4\theta) \right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8}
\end{aligned}
$$

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

Example

Find all points of intersection of the polar curves $r = \frac{1}{2}$ and $r = \cos(2\theta)$.

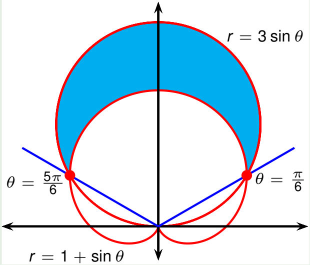


$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

- This only gives four points.
- There are actually eight.
- The circle $r = \frac{1}{2}$ also has polar equation $r = -\frac{1}{2}$.
- To find all eight points, solve $\cos(2\theta) = \frac{1}{2}$ and $\cos(2\theta) = -\frac{1}{2}$.

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2}\right) \\ &= \pi \end{aligned}$$