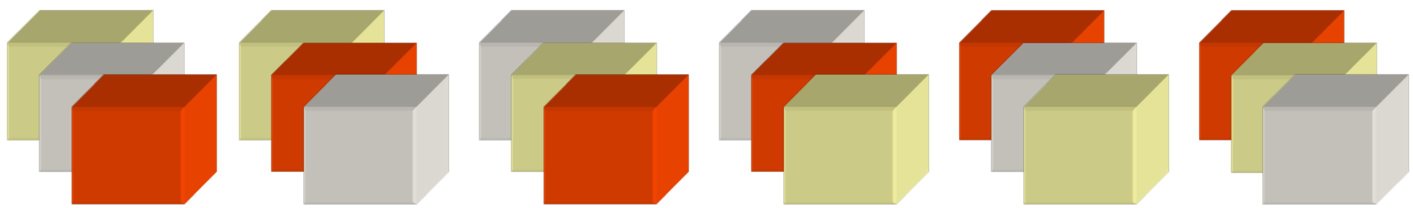
Binomial Coefficients and Pascal's triangle

CSC 1300 – Discrete Structures

Villanova University

Permutations

- A **permutation** is an ordering of objects
  - For example, 3 blocks can be ordered 6 ways

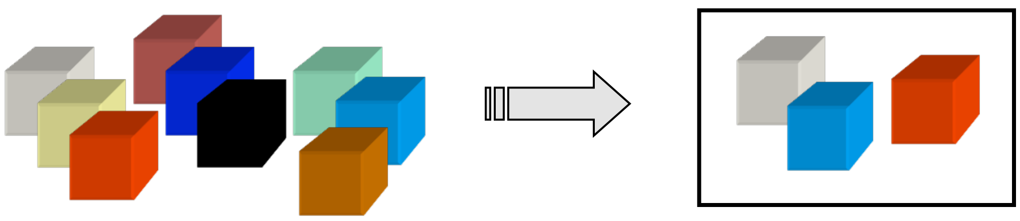


- There are n! permutations of n elements
  - Easily proved using the Product rule

k-Combination

What if all that matters is which blocks you select, not the order?...

- A **combination** is an *unordered* arrangement of elements in a set
  - Example: a 3-combination from a set of 12 colored blocks



Combinations

- **Choice notation:** ***n choose k***

$$\binom{n}{k}$$ = ***k-combinations of a set with n elements***

- also denoted: ***C(n,k)***
- The number of subsets of size k from a set with n elements
- Also called the ***binomial coefficient***
- Formula: C(n,k)= n!/[(n-k)!k!]

Combinations

- **Example:** the number of ways to form a committee of 4 members from a department of 13 faculty
- denoted C(13,4) or $\binom{13}{4}$

k-Permutations

P*(*n*,*k*)** = ***k-permutations of a set with n elements

- The number of ways to permute k out of n items
- Similar to combinations, but order matters
- Formula: $P(n,k) = n! / (n-k)!$

Permutations

- **Example:** the number of ways to arrange 4 of the 13 faculty to appear in a in a photo

- denoted   P(13,4)

Pascal's Triangle

- Pascal's Triangle represents the identity:

$$C(n+1,k) = C(n,k) + C(n,k\text{-}1) \qquad \text{for any } 1 \leq k \leq n$$

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

```
                    C(0,0)
               C(1,0)    C(1,1)
          C(2,0)    C(2 ,1)   C(2,2)
     C(3,0)    C(3,1)    C(3,2)    C(3,3)
C(4,0)    C(4,1)    C(4,2)    C(4,3)    C(4,4)
```

More Properties of Combinations

$C(n,k) = C(n, n\text{-}k)$ for any $0 \leq k \leq n$

$C(n,0) =$ for any $n \geq 0$

$C(n,n) =$

$C(n,0) + C(n,1) + \ldots + C(n,n) =$ for any $n \geq 0$

Binomial Refresher

- A binomial expression is simply the sum of two terms
  - For example:
    - $(x+y)$
    - $(x+y)^2$
- When a binomial expression is expanded, the binomial coefficients can be "seen"
  - For example:

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$= 1x^2 + 2xy + 1y^2$$

Binomial Coefficients & Combinations

- Explore the following:

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

$$x^3 + 3x^2y + 3xy^2 + y^3$$

C(3,3) C(3,2) C(3,1) C(3,0)

- Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k$$

Binomial Theorem

- <u>Problem</u>
  - What is the expansion of $(x+y)^4$?
- <u>Solution</u>
  - $(x+y)^4 = C(4,0)x^4y^0 +$
    $C(4,1)x^3y^1 +$
    $C(4,2)x^2y^2 +$
    $C(4,3)x^1y^3 +$
    $C(4,4)x^0y^4$

$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial Coefficients & Combinations

- <u>Problem</u>
  - Find the coefficient x^4y^7 in the expansion of $(x+y)^{11}$
- <u>Solution</u>

  n= 11 and k = 7

  $C(11,7)\ x^{11-7}\ y^7$

  $(11*10*9*8*7*6*5)/(7*6*5*4*3*2)\ x^{11-7}\ y^7$

  = **330** $x^4\ y^7$

Some Corollaries of the Binomial Theorem

Corollary 1 ($a = b = 1$):

Corollary 2 ($a = 1, b = -1$):

Corollary 3 ($a = 1, b = 2$):

bit strings to cut out for playing around with patterns

→ 00000	→ 00001	→ 00010	→ 00011
→ 00100	→ 00101	→ 00110	→ 00111
→ 01000	→ 01001	→ 01010	→ 01011
→ 01100	→ 01101	→ 01110	→ 01111
→ 10000	→ 10001	→ 10010	→ 10011
→ 10100	→ 10101	→ 10110	→ 10111
→ 11000	→ 11001	→ 11010	→ 11011
→ 11100	→ 11101	→ 11110	→ 11111

Below are rows zero to sixteen of Pascal's triangle in table form (even numbers highlighted): [source: wikipedia: http://en.wikipedia.org/wiki/Wikipedia_talk:WikiProject_Mathematics/Archive_37

row #	Pascal's triangle
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1
10	1 10 45 120 210 252 210 120 45 10 1
11	1 11 55 165 330 462 462 330 165 55 11 1
12	1 12 66 220 495 792 924 792 495 220 66 12 1
13	1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
14	1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
15	1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
16	1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1