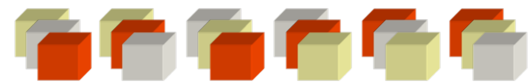
Binomial Coefficients and Pascal's triangle

CSC 1300 – Discrete Structures

Villanova University

Permutations

- A **permutation** is an ordering of objects
  - For example, 3 blocks can be ordered 6 ways

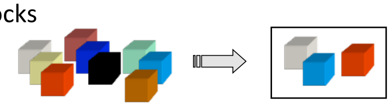


- There are n! permutations of n elements
  - Easily proved using the Product rule

k-Combination

What if all that matters is which blocks you select, not the order?...

- A **combination** is an *unordered* arrangement of elements in a set
  - Example: a 3-combination from a set of 12 colored blocks



Combinations

- **Choice notation:** ***n choose k***

$\binom{n}{k}$ = ***k-combinations of a set with n elements***

- also denoted: ***C(n,k)***
- The number of subsets of size k from a set with n elements
- Also called the ***binomial coefficient***
- Formula: C(n,k)= n!/[(n-k)!k!]

Combinations

- **<u>Example:</u>** the number of ways to form a committee of 4 members from a department of 13 faculty
- denoted C(13,4) or $\binom{13}{4}$

k-Permutations

P*(*n*,*k*)** = ***k-permutations of a set with n elements

- The number of ways to permute k out of n items
- Similar to combinations, but <u>order matters</u>
- Formula: P(n,k) = n! / (n-k)!

Permutations

- **<u>Example:</u>** the number of ways to arrange 4 of the 13 faculty to appear in a in a photo
- denoted P(13,4)

Pascal's Triangle

- Pascal's Triangle represents the identity:

$$C(n+1,k) = C(n,k) + C(n,k-1) \quad \text{for any } 1 \le k \le n$$

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        1                              C(0,0)
      1   1                        C(1,0)  C(1,1)
    1   2   1                  C(2,0)  C(2,1)  C(2,2)
  1   3   3   1            C(3,0)  C(3,1)  C(3,2)  C(3,3)
1   4   6   4   1      C(4,0)  C(4,1)  C(4,2)  C(4,3)  C(4,4)
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More Properties of Combinations

$C(n,k) = C(n, n-k)$ for any $0 \le k \le n$

$C(n,0) =$ for any $n \ge 0$

$C(n,n) =$

$C(n,0) + C(n,1) + \dots + C(n,n) =$ for any $n \ge 0$

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Binomial Refresher

- A binomial expression is simply the sum of two terms
  - For example:
    - $(x+y)$
    - $(x+y)^2$
- When a binomial expression is expanded, the binomial coefficients can be "seen"
  - For example:

$(x+y)^2 = x^2 + 2xy + y^2$
$= 1x^2 + 2xy + 1y^2$

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Binomial Coefficients & Combinations

- Explore the following:

$(x+y)^3 = (x+y)(x+y)(x+y)$
$xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$
$x^3 + 3x^2y + 3xy^2 + y^3$

C(3,3) C(3,2) C(3,1) C(3,0)

- Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^{n-k}y^k$$

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Binomial Theorem

- Problem
  - What is the expansion of $(x+y)^4$?
- Solution
  - $(x+y)^4 = C(4,0)x^4y^0 + C(4,1)x^3y^1 + C(4,2)x^2y^2 + C(4,3)x^1y^3 + C(4,4)x^0y^4$

$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

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Binomial Coefficients & Combinations

- <u>Problem</u>
  - Find the coefficient x^4y^7 in the expansion of $(x+y)^{11}$
- <u>Solution</u>

  n= 11 and k = 7

  C(11,7) x^{11-7} y^7

  (11*10*9*8*7*6*5)/(7*6*5*4*3*2) x^{11-7} y^7

  = **330** x^4 y^7

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Some Corollaries of the Binomial Theorem

Corollary 1 ($a = b = 1$):

Corollary 2 ($a = 1$, $b = -1$):

Corollary 3 ($a = 1$, $b = 2$):

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bit strings to cut out for playing around with patterns

➔ 00000	➔ 00001	➔ 00010	➔ 00011
➔ 00100	➔ 00101	➔ 00110	➔ 00111
➔ 01000	➔ 01001	➔ 01010	➔ 01011
➔ 01100	➔ 01101	➔ 01110	➔ 01111
➔ 10000	➔ 10001	➔ 10010	➔ 10011
➔ 10100	➔ 10101	➔ 10110	➔ 10111
➔ 11000	➔ 11001	➔ 11010	➔ 11011
➔ 11100	➔ 11101	➔ 11110	➔ 11111

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Below are rows zero to sixteen of Pascal's triangle in table form (even numbers highlighted): [source: wikipedia: http://en.wikipedia.org/wiki/Wikipedia_talk:WikiProject_Mathematics/Archive_37

row #	Pascal's triangle
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1
10	1 10 45 120 210 252 210 120 45 10 1
11	1 11 55 165 330 462 462 330 165 55 11 1
12	1 12 66 220 495 792 924 792 495 220 66 12 1
13	1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
14	1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
15	1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
16	1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

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