3. Symmetries and Conservation Laws

The dynamics of interacting field theories, such as ϕ^4 theory in Eq. (2.105), are extremely complex. The resulting equations of motion are not analytically soluble. In fact, free field theory (with the optional addition of a source term, as we will discuss) is the only field theory in four dimensions which has an analytic solution. Nevertheless, in more complicated interacting theories it is often possible to discover many important features about the solution simply by examining the symmetries of the theory. In this chapter we will look at this question in detail and develop some techniques which will allow us to extract dynamical information from the symmetries of a theory.

3.1 Classical Mechanics

Let's return to classical mechanics for a moment, where the Lagrangian is $L = T - V$. As a simple example, consider two particles in one dimension in a potential

$$L = \tfrac{1}{2}m_1\dot{q}_1^2 + \tfrac{1}{2}m_2\dot{q}_2^2 - V(q_1, q_2). \tag{3.1}$$

The momenta conjugate to the q_i's are $p_i = m_i\dot{q}_i$, and from the Euler-Lagrange equations

$$\dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad \dot{P} \equiv \dot{p}_1 + \dot{p}_2 = -\left(\frac{\partial V}{\partial q_1} + \frac{\partial V}{\partial q_2}\right). \tag{3.2}$$

If V depends only on $q_1 - q_2$ (that is, the particles aren't attached to springs or anything else which defines a fixed reference frame) then the system is invariant under the shift $q_i \to q_i + \alpha$, and $\partial V/\partial q_1 = -\partial V/\partial q_2$, so $\dot{P} = 0$. The total momentum of the system is conserved. A symmetry ($L(q_i + \alpha, \dot{q}_i) = L(q_i, \dot{q}_i)$) has resulted in a conservation law.

We also saw earlier that when $\partial L/\partial t = 0$ (that is, L depends on t only through the coordinates q_i and their derivatives), then $dH/dt = 0$. H (the energy) is therefore a conserved quantity when the system is invariant under time translation.

This is a very general result which goes under the name of Noether's theorem: for every symmetry, there is a corresponding conserved quantity. It is useful because it allows you to make exact statements about the solutions of a theory without solving it explicitly. Since in quantum field theory we won't be able to solve anything exactly, symmetry arguments will be extremely important.

To prove Noether's theorem, we first need to define "symmetry." Given some general transformation $q_a(t) \to q_a(t, \lambda)$, where $q_a(t, 0) = q_a(t)$, define

$$Dq_a \equiv \left.\frac{\partial q_a}{\partial \lambda}\right|_{\lambda=0} \tag{3.3}$$

For example, for the transformation $\vec{r} \to \vec{r} + \lambda\hat{e}$ (translation in the $\hat{e}$ direction), $D\vec{r} = \hat{e}$. For time translation, $q_a(t) \to q_a(t + \lambda) = q_a(t) + \lambda dq_a/dt + \mathcal{O}(\lambda^2)$, $Dq_a = dq_a/dt$.

You might imagine that a symmetry is defined to be a transformation which leaves the Lagrangian invariant, $DL = 0$. Actually, this is too restrictive. Time translation, for example, doesn't satisfy this requirement: if L has no explicit t dependence,

$$L(t, \lambda) = L(q_a(t + \lambda), \dot{q}_a(t + \lambda)) = L(0) + \lambda\frac{dL}{dt} + ... \tag{3.4}$$

so $DL = dL/dt$. So more generally, a transformation is a symmetry iff $DL = dF/dt$ for some function $F(q_a, \dot{q}_a, t)$. Why is this a good definition? Consider the variation of the action S:

$$DS = \int_{t_1}^{t_2} dt DL = \int_{t_1}^{t_2} dt \frac{dF}{dt} = F(q_a(t_2), \dot{q}_a(t_2), t_2) - F(q_a(t_1), \dot{q}_a(t_1), t_1). \tag{3.5}$$

Recall that when we derived the equations of motion, we didn't vary the q_a's and $\dot{q}_a$'s at the endpoints, $\delta q_a(t_1) = \delta q_a(t_2) = 0$. Therefore the additional term doesn't contribute to δS and therefore doesn't affect the equations of motion.

It is now easy to prove Noether's theorem by calculating DL in two ways. First of all,

$$\begin{aligned} DL &= \sum_a \frac{\partial L}{\partial q_a} Dq_a + \frac{\partial L}{\partial \dot{q}_a} D\dot{q}_a \\ &= \sum_a \dot{p}_a Dq_a + p_a D\dot{q}_a \\ &= \frac{d}{dt} \sum_a p_a Dq_a \end{aligned} \tag{3.6}$$

where we have used the equations of motion and the equality of mixed partials ($D\dot{q}_a = d(Dq_a)/dt$). But by the definition of a symmetry, $DL = dF/dt$. So

$$\frac{d}{dt}\left(\sum_a p_a Dq_a - F \right) = 0. \tag{3.7}$$

So the quantity $\sum_a p_a Dq_a - F$ is conserved.

Let's apply this to our two previous examples.

1. Space translation: $q_i \to q_i + \alpha$. Then $DL = 0$, $p_i = m_i \dot{q}_i$ and $Dq_i = 1$, so $p_1 + p_2 = m_1 \dot{q}_1 + m_2 \dot{q}_2$ is conserved. We will call any conserved quantity associated with spatial translation invariance momentum, even if the system looks nothing like particle mechanics.

2. Time translation: $t \to t + \lambda$. Then $Dq_a = dq_a/dt$, $DL = dL/dt$, $F = L$ and so the conserved quantity is $\sum_a (p_a \dot{q}_a) - L$. This is the Hamiltonian, justifying our previous assertion that the Hamiltonian is the energy of the system. Again, we will call the conserved quantity associated with time translation invariance the energy of the system.

This works for classical particle mechanics. Since the canonical commutation relations are set up to reproduce the E-L equations of motion for the operators, it will work for quantum particle mechanics as well.

3.2 Symmetries in Field Theory

Since field theory is just the continuum limit of classical particle mechanics, the same arguments must go through as well. In fact, stronger statements may be made in field theory, because not only are conserved quantities globally conserved, they must be locally conserved as well. For example, in a theory which conserves electric charge we can't have two separated opposite charges simultaneously wink out of existence. This conserves charge globally, but not locally. Recall from electromagnetism that the charge density satisfies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\jmath} = 0. \tag{3.8}$$

This just expresses current conservation. Integrating over some volume V, and defining $Q_V = \int_V d^3x \rho(x)$, we have

$$\frac{dQ_V}{dt} = -\int_v \nabla \cdot \vec{\jmath} = -\int_S dS \cdot \vec{\jmath} \tag{3.9}$$

where S is the surface of V. This means that the rate of change of charge inside some region is given by the flux through the surface. Taking the surface to infinity, we find that the total charge Q is conserved. However, we have the stronger statement of current conservation, Eq. (3.8). Therefore, in field theory conservation laws will be of the form $\partial_\mu \mathcal{J}^\mu = 0$ for some four-current $\mathcal{J}^\mu$.

As before, we consider the transformations $\phi_a(x) \to \phi_a(x, \lambda)$, $\phi_a(x, 0) = \phi_a(x)$, and define

$$D\phi_a = \frac{\partial \phi_a}{\partial \lambda}\bigg|_{\lambda=0}. \tag{3.10}$$

A transformation is a symmetry iff $D\mathcal{L} = \partial_\mu F^\mu$ for some $F^\mu(\phi_a, \dot{\phi}_a, x)$. I will leave it to you to show that, just as in particle mechanics, a transformation of this form doesn't affect the equations of motion. We now have

$$\begin{aligned} D\mathcal{L} &= \sum_a \frac{\partial L}{\partial \phi_a} D\phi_a + \Pi^\mu_a D(\partial_\mu \phi_a) \\ &= \sum_a \partial_\mu \Pi^\mu_a D\phi_a + \Pi^\mu_a \partial_\mu D\phi_a \\ &= \partial_\mu \sum_a (\Pi^\mu_a D\phi_a) = \partial_\mu F^\mu \end{aligned} \tag{3.11}$$

so the four components of

$$J^\mu = \sum_a \Pi^\mu_a D\phi_a - F^\mu \tag{3.12}$$

satisfy $\partial_\mu J^\mu = 0$. If we integrate over all space, so that no charge can flow out through the boundaries, this gives the global conservation law

$$\frac{dQ}{dt} \equiv \frac{d}{dt} \int d^3x J^0(x) = 0. \tag{3.13}$$

3.2.1 Space-Time Translations and the Energy-Momentum Tensor

We can use the techniques from the previous section to calculate the conserved current and charge in field theory corresponding to a space or time translation. Under a shift $x \to x + \lambda e$, where e is some fixed four-vector, we have

$$\begin{aligned} \phi_a(x) &\to \phi_a(x + \lambda e) \\ &= \phi_a(x) + \lambda e_\mu \partial^\mu \phi_a(x) + ... \end{aligned} \tag{3.14}$$

so

$$D\phi_a(x) = e_\mu \partial^\mu \phi_a(x). \tag{3.15}$$

Similarly, since $\mathcal{L}$ contains no explicit dependence on x but only depends on it through the fields ϕ^a, we have $D\mathcal{L} = \partial_\mu(e^\mu \mathcal{L})$, so $F = e^\mu \mathcal{L}$. The conserved current is therefore

$$\begin{aligned} J^\mu &= \sum_a \Pi_a^\mu D\phi - F \\ &= \sum_a \Pi_a^\mu e_\nu \partial^\nu \phi_a - e^\mu \mathcal{L} \\ &= e_\nu \left[\sum_a \Pi_a^\mu \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L} \right] \\ &\equiv e_\nu T^{\mu\nu} \end{aligned} \tag{3.16}$$

where $T^{\mu\nu} = \sum_a \Pi_a^\mu \partial^\nu \phi_a - g^{\mu\nu}\mathcal{L}$ is called the *energy-momentum tensor*. Since $\partial_\mu J^\mu = 0 = \partial_\mu T^{\mu\nu} e_\nu$ for arbitrary e, we also have

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.17}$$

For time translation, $e_\nu = (1, \vec{0})$. $T^{\mu 0}$ is therefore the "energy current", and the corresponding conserved quantity is

$$Q = \int d^3x J^0 = \int d^3x T^{00} = \int d^3x \sum_a \left(\Pi_a^0 \partial_0 \phi_a - \mathcal{L} \right) = \int d^3x \mathcal{H} \tag{3.18}$$

where $\mathcal{H}$ is the Hamiltonian density we had before. So the Hamiltonian, as we had claimed, really is the energy of the system (that is, it corresponds to the conserved quantity associated with time translation invariance.)

Similarly, if we choose $e^\mu = (0, \hat{x})$ then we will find the conserved charge to be the x-component of momentum. For the Klein-Gordon field, a straightforward substitution of the expansion of the fields in terms of creation and annihilation operators into the expression for $\int d^3x T^{01}$ gives the expression we obtained earlier for the momentum operator,

$$: \vec{P} := \int d^3k \, \vec{k} \, a_k^\dagger a_k \tag{3.19}$$

where again we have normal-ordered the expression to remove spurious infinities.

Note that the *physical* momentum $\vec{P}$, the conserved charge associated with space translation, has nothing to do with the *conjugate* momentum Π_a of the field ϕ_a. It is important not to confuse these two uses of the term "momentum."

3.2.2 Lorentz Transformations

Under a Lorentz transformation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{3.20}$$

a four-vector transforms as

$$a^\mu \to \Lambda^\mu{}_\nu a^\nu \tag{3.21}$$

as discussed in the first section. Since a scalar field by definition does not transform under Lorentz transformations, it has the simple transformation law

$$\phi(x) \to \phi(\Lambda^{-1}x). \tag{3.22}$$

This simply states that the field itself does not transform at all; the value of the field at the coordinate x in the new frame is the same as the field at that same point in the old frame. Fields with spin have more complicated transformation laws, since the various components of the fields rotate into one another under Lorentz transformations. For example, a vector field (spin 1) A_μ transforms as

$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{3.23}$$

As usual, we will restrict ourselves to scalar fields at this stage in the course.

To use the machinery of the previous section, let us consider a one parameter subgroup of Lorentz transformations parameterized by λ. This could be rotations about a specified axis by an angle λ, or boosts in some specified direction with $\gamma = \lambda$. This will define a family of Lorentz transformations $\Lambda(\lambda)^\mu{}_\nu$, from which we wish to get $D\phi = \partial\phi/\partial\lambda|_{\lambda=0}$. Let us define

$$\epsilon^\mu{}_\nu \equiv D\Lambda^\mu{}_\nu. \tag{3.24}$$

Then under a Lorentz transformation $a^\mu \to \Lambda^\mu{}_\nu a^\nu$, we have

$$Da^\mu = \epsilon^\mu{}_\nu a^\nu. \tag{3.25}$$

It is straightforward to show that $\epsilon_{\mu\nu}$ is antisymmetric. From the fact that $a^\mu b_\mu$ is Lorentz invariant, we have

$$\begin{aligned} 0 &= D(a^\mu b_\mu) = (Da^\mu)b_\mu + a^\mu(Db_\mu) \\ &= \epsilon^\mu{}_\nu a^\nu b_\mu + a^\mu \epsilon_\mu{}^\nu b_\nu \\ &= \epsilon_{\mu\nu} a^\nu b^\mu + \epsilon_{\nu\mu} a^\nu b^\mu \\ &= (\epsilon_{\mu\nu} + \epsilon_{\nu\mu})\, a^\nu b^\mu \end{aligned} \tag{3.26}$$

where in the third line we have relabelled the dummy indices. Since this holds for arbitrary four vectors a^μ and b^ν, we must have

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \tag{3.27}$$

The indices μ and ν range from 0 to 3, which means there are $4(4-1)/2 = 6$ independent components of ϵ. This is good because there are six independent Lorentz transformations - three rotations (one about each axis) and three boosts (one in each direction).

Let's take a moment and do a couple of examples to demystify this. Take $\epsilon_{12} = -\epsilon_{21}$ and all the other components zero. Then we have

$$\begin{aligned} Da^1 &= \epsilon^1_2 a^2 = -\epsilon_{12} a^2 = -a^2 \\ Da^2 &= \epsilon^2_1 a^1 = -\epsilon_{21} a^2 = +a^1. \end{aligned} \tag{3.28}$$

This just corresponds to the a rotation about the z axis,

$$\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \to \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}. \tag{3.29}$$

On the other hand, taking $\epsilon_{01} = -\epsilon_{10} = +1$ and all other components zero, we get

$$\begin{aligned} Da^0 &= \epsilon^0_1 a^1 = \epsilon_{01} a^1 = +a^1 \\ Da^1 &= \epsilon^1_0 a^1 = -\epsilon_{10} a^2 = +a^0. \end{aligned} \tag{3.30}$$

Note that the signs are different because lowering a 0 index doesn't bring in a factor of -1. This is just the infinitesimal version of

$$\begin{pmatrix} a^0 \\ a^1 \end{pmatrix} \to \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \end{pmatrix}. \tag{3.31}$$

which corresponds to a boost along the x axis.

Now we're set to construct the six conserved currents corresponding to the six different Lorentz transformations. Using the chain rule, we find

$$\begin{aligned} D\phi(x) &= \frac{\partial}{\partial\lambda}\, \phi(\Lambda^{-1}\,(\lambda)^\mu{}_\nu x^\nu)\big|_{\lambda=0} \\ &= \partial_\alpha\phi(x)\, \frac{\partial}{\partial\lambda}\, (\Lambda^{-1}(\lambda)(x)^\alpha\big|_{\lambda=0} \\ &= \partial_\alpha\phi(x)\, D\left(\Lambda^{-1}(\lambda)^\alpha{}_\beta x^\beta\right) \\ &= \partial_\alpha\phi(x)\left(-\epsilon^\alpha_\beta\right) x^\beta \\ &= -\epsilon_{\alpha\beta} x^\beta \partial^\alpha\phi(x). \end{aligned} \tag{3.32}$$

Since $\mathcal{L}$ is a scalar, it depends on x only through its dependence on the field and its derivatives. Therefore we have

$$\begin{aligned} D\mathcal{L} &= \epsilon_{\alpha\beta} x^\alpha \partial^\beta \mathcal{L} \\ &= \partial_\mu\left(\epsilon_{\alpha\beta} x^\beta g^{\mu\alpha}\right)\mathcal{L} \end{aligned} \tag{3.33}$$

and so the conserved current J^μ is

$$\begin{aligned} J^\mu &= \sum_a \left(\Pi^\mu \epsilon_{\alpha\beta} x^\alpha \partial^\beta\phi - \epsilon_{\alpha\beta} x^\alpha g^{\mu\beta}\mathcal{L}\right) \\ &= \epsilon_{\alpha\beta}\left(\Pi^\mu x^\alpha \partial^\beta\phi - x^\alpha g^{\mu\beta}\mathcal{L}\right). \end{aligned} \tag{3.34}$$

Since the current must be conserved for all six antisymmetric matrices $\epsilon_{\alpha\beta}$, the part of the quantity in the parentheses that is antisymmetric in α and β must be conserved. That is,

$$\partial_\mu M^{\mu\alpha\beta} = 0 \tag{3.35}$$

where

$$\begin{aligned} M^{\mu\alpha\beta} &= \Pi^\mu x^\alpha \partial^\beta\phi - x^\alpha g^{\mu\beta}\mathcal{L} - \alpha \leftrightarrow \beta \\ &= x^\alpha\left(\Pi^\mu \partial^\beta\phi - g^{\mu\beta}\mathcal{L}\right) - \alpha \leftrightarrow \beta \\ &= x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} \end{aligned} \tag{3.36}$$

where $T^{\mu\nu}$ is the energy-momentum tensor defined in Eq. (3.16). The six conserved charges are given by the six independent components of

$$J^{\alpha\beta} = \int d^3x \, M^{0\alpha\beta} = \int d^3x \, \left(x^\alpha T^{0\beta} - x^\beta T^{0\alpha} \right). \tag{3.37}$$

Just as we called the conserved quantity corresponding to space translation the momentum, we will call the conserved quantity corresponding to rotations the angular momentum. So for example J^{12}, the conserved quantity coming from invariance under rotations about the 3 axis, is

$$J^{12} = \int d^3x \, \left(x^1 T^{02} - x^2 T^{01} \right). \tag{3.38}$$

This is the field theoretic analogue of angular momentum. We can see that this definition matches our previous definition of angular momentum in the case of a point particle with position $\vec{r}(t)$. In this case, the energy momentum tensor is

$$T^{0i}(\vec{x}, t) = p^i \delta^{(3)}(\vec{x} - \vec{r}(t)) \tag{3.39}$$

which gives

$$J^{12} = x^1 p^2 - x^2 p^1 = (\vec{r} \times \vec{p})_3 \tag{3.40}$$

which is the familiar expression for the third component of the angular momentum. Note that this is only for scalar particles. Particles with spin carry intrinsic angular momentum which is not included in this expression - this is only the orbital angular momentum. Particles with spin are described by fields with tensorial character, which is reflected by additional terms in the J^{ij}.

That takes care of three of the invariants corresponding to Lorentz transformations. Together with energy and linear momentum, they make up the conserved quantities you learned about in first year physics. What about boosts? There must be three more conserved quantities. What are they? Consider

$$J^{0i} = \int d^3x \, \left(x^0 T^{0i} - x^i T^{00} \right). \tag{3.41}$$

This has an explicit reference to x^0, the time, which is something we haven't seen before in a conservation law. But there's nothing in principle wrong with this. The x^0 may be pulled out of the spatial integral, and the conservation law gives

$$\begin{aligned} 0 &= \frac{d}{dt} J^{0i} = \frac{d}{dt} \left[t \int d^3x \, T^{0i} - \int d^3x \, x^i T^{00} \right] \\ &= t \frac{d}{dt} \int d^3x \, T^{0i} + \int d^3x \, T^{0i} - \frac{d}{dt} \int d^3x \, x^i T^{00} \\ &= t \frac{d}{dt} p^i + p^i - \frac{d}{dt} \int d^3x \, x^i T^{00}. \end{aligned} \tag{3.42}$$

The first term is zero by momentum conservation, and the second term, p^i, is a constant. Therefore we get

$$p^i = \frac{d}{dt} \int d^3x \, x^i T^{00} = \text{constant}. \tag{3.43}$$

This is just the field theoretic and relativistic generalization of the statement that the centre of mass moves with a constant velocity. The centre of mass is replaced by the "centre of energy." Although you are not used to seeing this presented as a separate conservation law from conservation of momentum, we see that in field theory the relation between the T^{0i}'s and the first moment of T^{00} is the result of Lorentz invariance. The three conserved quantities $\int d^3x\, x^i T^{00}(x)$ are the Lorentz partners of the angular momentum.

3.3 Internal Symmetries

Energy, momentum and angular momentum conservation are clearly properties of any Lorentz invariant field theory. We could write down an expression for the energy-momentum tensor $T^{\mu\nu}$ without knowing the explicit form of $\mathcal{L}$. However, there are a number of other quantities which are experimentally known to be conserved, such as electric charge, baryon number and lepton number which are not automatically conserved in any field theory. By Noether's theorem, these must also be related to continuous symmetries. Experimental observation of these conservation laws in nature is crucial in helping us to figure out the Lagrangian of the real world, since they require $\mathcal{L}$ to have the appropriate symmetry and so tend to greatly restrict the form of $\mathcal{L}$. We will call these transformations which don't correspond to space-time transformations internal symmetries.

3.3.1 $U(1)$ Invariance and Antiparticles

Here is a theory with an internal symmetry:

$$\mathcal{L} = \tfrac{1}{2}\sum_{a=1}^{2} \partial_\mu\phi_a\partial^\mu\phi_a - \mu^2\phi_a\phi_a - g\left(\sum_a (\phi_a)^2\right)^2 . \tag{3.44}$$

It is a theory of two scalar fields, ϕ_1 and ϕ_2, with a common mass μ and a potential $g\left(\sum_a(\phi_a)^2\right)^2 = g\left((\phi_1)^2 + (\phi_2)^2\right)^2$. This Lagrangian is invariant under the transformation

$$\begin{aligned}\phi_1 &\to \phi_1\cos\lambda + \phi_2\sin\lambda\\ \phi_2 &\to -\phi_1\sin\lambda + \phi_2\cos\lambda.\end{aligned} \tag{3.45}$$

This is just a rotation of ϕ_1 into ϕ_2 in field space. It leaves $\mathcal{L}$ invariant (try it) because $\mathcal{L}$ depends only on $\phi_1^2+\phi_2^2$ and $(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2$, and just as $r^2 = x^2+y^2$ is invariant under real rotations, these are invariant under the transformation (3.45).

We can write this in matrix form:

$$\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = \begin{pmatrix}\cos\lambda & \sin\lambda\\ -\sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{3.46}$$

In the language of group theory, this is known as an $SO(2)$ transformation. The S stands for "special", meaning that the transformation matrix has unit determinant, the O for "orthogonal" and the 2 because it's a 2×2 matrix. We say that $\mathcal{L}$ has an $SO(2)$ symmetry.

Once again we can calculate the conserved charge:

$$\begin{aligned}D\phi_1 &= \phi_2\\ D\phi_2 &= -\phi_1\\ D\mathcal{L} &= 0 \to F^\mu = \text{constant}.\end{aligned} \tag{3.47}$$