1. Explicitly verify Wick's theorem for the case of three scalar fields:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) = :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)D_F(x_2 - x_3) + \phi(x_2)D_F(x_3 - x_1) + \phi(x_3)D_F(x_1 - x_2). \quad (1)$$

2. Let us return to the problem of the creation of Klein-Gordon particles by a classical source. Recall that this process can be described by the Hamiltonian

$$H = H_0 + \int d^3x\,(-\rho(t,\vec{x})\phi(x))$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $\rho(x)$ is a c-number (*i.e.* classical number, not an operator) scalar function. We found that, if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{\rho}(p)|^2$$

where

$$\tilde{\rho}(p) = \int d^4y\, e^{ip\cdot y}\rho(y)$$

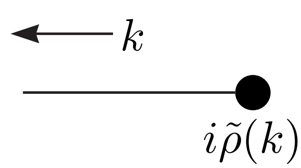
evaluated at 4-momenta p such that $p^2 = m^2$. In this problem we will solve this theory using the perturbation techniques we have developed, verifying the result above for the mean number of particles, and extracting more detailed information about the final state.

(a) Show that the probability that the source creates *no* particles is given by

$$P(0) = \left|\langle 0|T\left\{\exp[i\int d^4x\rho(x)\phi_I(x)]\right\}|0\rangle\right|^2.$$

(b) Now let us do perturbation theory in powers of ρ. Evaluate the term in $P(0)$ of order ρ^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(\rho^4)$, where λ equals the expression given above for $\langle N\rangle$.

(c) Show that you can represent the term computed in part (b) as a Feynman diagram, where the Feynman rules for the theory include the usual propagator as well as the vacuum-to-one-particle interaction below.



(Note that this is what you should expect by looking at the interaction Hamiltonian; it only contains a single field, which may either create or annihilate a single meson. The amplitude to create a meson of momentum k is proportional to the corresponding Fourier component of the source).

(d) Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly:

$$P(0) = \exp(-\lambda).$$

(e) Compute the probability that the source creates one particle of momentum k. Perform this computation first to $\mathcal{O}(\rho)$ and then to all orders, using the trick of part (d) to sum the series.

(f) Show that the probability of producing n particles is given by

$$P(n) = (1/n!)\lambda^n \exp(-\lambda).$$

This is a *Poisson distribution.* (NOTE: The combinatorics get a bit nasty here. If you can't do the general case right off the bat, first try it for the first few values of n and look for the pattern.)

(g) Prove the following facts about a Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1; \quad \langle N \rangle = \sum_{n=0}^{\infty} nP(n) = \lambda.$$

The first identity says that the $P(n)$'s are properly normalized probabilities, while the second confirms our previous result for $\langle N \rangle$. Compute the mean square fluctuation $\langle (N - \langle N \rangle)^2 \rangle$.