





Almost constant at "low" temperatures...all way to linear at high temperatures





- Increases with impurity content
- Has some "universal" features...



### Our Immediate Questions...and Answers

- What is resistivity anyway? Linear response..
- "Simplest" ideas about resistivity
  - Drudé theory
  - Bloch-Boltzmann theory
- Crux: "Semi" classical ideas, mean free path, relaxation time...

#### What are we measuring in experiments?

- The conductivity tensor response function in nice systems can be written in Fourier space  $\sigma(q, \omega)$
- The complex amplitude  $j(q, \omega)$  of the current response for an electric field  $E(r, t) = E(q, \omega)e^{i(q \cdot r - \omega t)}$  is given as  $j(q, \omega) = \sigma(q, \omega) \cdot E(q, \omega)$
- Imagine  $q \longrightarrow 0, \omega \longrightarrow 0$ , i.e., a "constant" electric field; the response is described be the complex tensor  $\sigma(q \rightarrow 0, \omega \rightarrow 0)$  (note: order of limit is crucial, more later)
- Assume isotropic system, then  $\sigma$  (conductivity) is simply  $\Re \sigma(\mathbf{q} \rightarrow \mathbf{0}, \omega \rightarrow 0)$
- **P** Resistivity  $\rho = 1/\sigma!$



## Drudé Theory – Review

- Electrons: a classical gas
- **Solution** Solution au and au, gives the equation of motion

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\boldsymbol{p}}{\tau} + \boldsymbol{F}$$

p – momentum, F – "external" force

Gives the "standard result" for conductivity

$$\sigma = \frac{ne^2\tau}{m}$$

(all symbols have usual meanings)

All is, however, not well with Drudé theory!



# **Bloch Theory**

- We do need quantum mechanics to understand metals (all materials, in fact)
- In the periodic potential of the ions, wave functions are  $\psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_k(\mathbf{r})$  ( $u_k$  is a lattice periodic function),  $\mathbf{k}$  is a vector in the 1st Brillouin zone
- The Hamiltonian expressed in Bloch language  $H = \sum_{k\sigma} \varepsilon(k) |k\rangle \langle k|$  (one band),  $\varepsilon(k)$  is the band dispersion (set aside spin throughout these lectures!)

• "Average velocity" in a Bloch state  $v(k) = \frac{1}{\hbar} \frac{\partial \varepsilon}{\partial k}$ 

• Occupancy of a Bloch state  $f^0(\mathbf{k}) = \frac{1}{e^{\beta(\varepsilon(\mathbf{k})-\mu)}+1}$ ,  $\beta = 1/(k_B T)$ ,  $\mu$  – chemical potential



# So, what is a metal?

- Chemical potential  $\mu$  determined from electron concentration
- **•** Try to construct a surface in the reciprocal space such that  $\varepsilon(\mathbf{k}) = \mu$
- If such a surface exists (at T = 0) we say that the material is a *metal*
- **A metal has a** *Fermi surface*
- Ok, so how do we calculate conductivity?
- Need to understand "how electron moves" under the action of "external forces"



## Semi-classical Electron Dynamics

- Key idea: External forces (F; electric/magnetic fields) cause transition of electronic states
- Rate of transitions  $\hbar \frac{\mathrm{d} k}{\mathrm{d} t} = F Quantum \text{ version of "Newton's law"$
- By simple algebra, we see the "acceleration"  $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{M}^{-1} \boldsymbol{F}, \ \boldsymbol{M}^{-1} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon}{\partial \boldsymbol{k} \partial \boldsymbol{k}}$
- Electron becomes a "new particle" in a periodic potential! Properties determined by value of M at the chemical potential
- But, what about conductivity? If you think about this, you will find a very surprising result! (Essentially infinite!)



## **Conductivity in Metals**

- What makes for *finite* conductivity in metals?
- Answer: "Collisions"
- Electrons may scatter from impurities/defects, electron-electron interactions, electron-phonon interaction etc...
- How do we model this? Brute force approach of solving the full Schrödinger equation is highly impractical!
- ✓ Key idea: The electron gets a "life-time" i.e., an electron placed in a Bloch state k evolves according to  $\psi(t) \sim \psi_k e^{-i\varepsilon(k)t \frac{t}{2\tau_k}}; \text{ "lifetime" is } \tau_k!$
- Source Conductivity could plausibly be related to  $\tau_k$ ; how?



# **Boltzmann Theory**

- **•** Nonequilibrium distribution function  $f(\boldsymbol{r}, \boldsymbol{k}, t)$ :
  - "Occupancy" of state k at position r and time t
  - r in f(r, k, t) represents a suitable "coarse grained" length scale (much greater than the atomic scale) such that "each" r represents a thermodynamic system
- Idea 1: The (possibly nonequilibrium) state of a system is described by a distribution function f(r, k, t)
- Idea 3: Collisions act to "restore" equilibrium try to bring f back to  $f^0$



# Time Evolution of $f(\boldsymbol{r}, \boldsymbol{k}, t)$

- Suppose we know f at time t = 0, what will it be at a later time t if we know all the "forces" acting on the system?
- Use semi-classical dynamics: An electron at r in state k at time t was at  $r v\Delta t$  in the state  $k \frac{F}{\hbar}\Delta t$  at time  $t \Delta t$
- Thus, we get the Boltzmann transport equation  $f(\boldsymbol{r}, \boldsymbol{k}, t) = f(\boldsymbol{r} - \boldsymbol{v}\Delta t, \boldsymbol{k} - \frac{\boldsymbol{F}}{\hbar}\Delta t, t - \Delta t) + \frac{\partial f}{\partial t}\Big|_{coll.} \Delta t$   $\implies \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} = \frac{\partial f}{\partial t}\Big|_{coll.}$
- If we specify the forces and the collision term, we have an initial value problem to determine f(r, k, t)



# **Electrical Conductivity**

#### BTE becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} = -\frac{f - f^0}{\tau_{\boldsymbol{k}}}$$

**•** Homogeneous DC electric field F = -eE

We look for the steady homogeneous response

$$\frac{\boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}} = -\frac{f - f^0}{\tau_{\boldsymbol{k}}} \implies f = f^0 - \frac{\tau_{\boldsymbol{k}} \boldsymbol{F}}{\hbar} \cdot \frac{\partial f}{\partial \boldsymbol{k}}$$

• Approximate solution (Exercise: Work this out)  

$$f(\mathbf{k}) \approx f^0 + \frac{e\tau_{\mathbf{k}}\mathbf{E}}{\hbar} \cdot \frac{\partial f^0}{\partial \mathbf{k}} \approx f^0 \left(\mathbf{k} + \frac{e\tau_{\mathbf{k}}\mathbf{E}}{\hbar}\right)$$



#### Solution of BTE



Fermi surface "shifts" (Exercise: estimate order of magnitude of shift)



# **Conductivity from BTE**

#### Current

$$\boldsymbol{j} = \frac{1}{(2\pi)^3} \int d^3 \boldsymbol{k} \left(-e\boldsymbol{v}\right) \frac{e\tau_{\boldsymbol{k}}\boldsymbol{E}}{\hbar} \cdot \frac{\partial f^0}{\partial \boldsymbol{k}}$$

Conductivity tensor

$$\boldsymbol{\sigma} = -\frac{1}{(2\pi)^3} \frac{e^2}{\hbar} \int d^3 \boldsymbol{k} \ \tau_{\boldsymbol{k}} \ \boldsymbol{v} \ \frac{\partial f^0}{\partial \boldsymbol{k}}$$

Further, with spherical Fermi-surface (free electron like),  $\tau_k$  roughly independent of k (Exercise: Show this)

$$\boldsymbol{\sigma} = \frac{ne^2\tau}{m} \mathbf{1}$$

This looks strikingly close to the Drudé result, but thephysics could not be more different!VBSFermions in Disorder - 17



## What about experiments?

- Well, we now have an expression for conductivity; we should compare with experiments?
- What determines the T dependence of conductivity?
   Yes, it is essentially the T dependence of τ (only in metals)
- **But we do not yet have**  $\tau$ **!!**
- **•** Need a way to calculate  $\tau$ ...
- **\_**
- Revisit the idea of electron-lifetime...how do we calculate life time of an electron?



### Lifetime due to Impurity Scattering

- Impurity potential V<sub>I</sub>, causes transitions from one Bloch state to another
- Sate of transition from k 
  ightarrow k'

$$W_{\boldsymbol{k}\to\boldsymbol{k}'} = \frac{2\pi}{\hbar} |\langle \boldsymbol{k}' | V_I | \boldsymbol{k} \rangle|^2 \delta(\varepsilon(\boldsymbol{k}') - \varepsilon(\boldsymbol{k}))$$

Total rate of transition, or inverse lifetime

$$\frac{1}{\tau_{\boldsymbol{k}}^{I}} = \frac{1}{(2\pi)^{3}} \int \mathrm{d}^{3}\boldsymbol{k}' \, W_{\boldsymbol{k} \rightarrow \boldsymbol{k}'}$$

- **9** Can we use  $\tau_{k}^{I}$  as the au in the Boltzmann equation?
- Ok in order of magnitude, but not alright! Why?



### How to calculate $\tau$ ?

Look back at the collision term, can write it more elaborately as

$$\frac{\partial f}{\partial t}\Big|_{coll.} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 \mathbf{k}' W_{\mathbf{k}\to\mathbf{k}'} \left(f(\mathbf{k})(1-f(\mathbf{k}')) - f(\mathbf{k}')(1-f(\mathbf{k}))\right)$$
$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 \mathbf{k}' W_{\mathbf{k}\to\mathbf{k}'} \left(f(\mathbf{k}) - f(\mathbf{k}')\right)$$

Note that k and k' are of the same energy

**•** Take  $\tau_{k}$  to depend only on  $\varepsilon(k)$ 



### **Calculation of** $\tau$ **cont'd**

#### Putting it all together

$$-\frac{e}{\hbar}\frac{\partial f^{0}}{\partial \varepsilon}\boldsymbol{v}(\boldsymbol{k})\cdot\boldsymbol{E} = -\frac{1}{(2\pi)^{3}}\frac{\tau e}{\hbar}\frac{\partial f^{0}}{\partial \varepsilon}\int \mathrm{d}^{3}\boldsymbol{k}' W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(\boldsymbol{v}(\boldsymbol{k})-\boldsymbol{v}(\boldsymbol{k}')\right) + \\ \Longrightarrow \frac{1}{\tau} = \frac{1}{(2\pi)^{3}}\int \mathrm{d}^{3}\boldsymbol{k}' W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(1-\frac{\boldsymbol{v}(\boldsymbol{k}')\cdot\hat{\boldsymbol{E}}}{\boldsymbol{v}(\boldsymbol{k})\cdot\hat{\boldsymbol{E}}}\right) \\ \Longrightarrow \frac{1}{\tau} = \frac{1}{(2\pi)^{3}}\int \mathrm{d}^{3}\boldsymbol{k}' W_{\boldsymbol{k}\to\boldsymbol{k}'}\left(1-\cos\left(\widehat{\boldsymbol{k},\boldsymbol{k}'}\right)\right)$$

**•** Note  $\tau$  is *different* from the "quasiparticle" life time!

Several Sev



# T dependence of $\tau$

- *T* dependence strongly depends on the mechanism of scattering
- Common scattering mechanisms
  - Impurity scattering
  - e-e scattering  $\sim T^2$
  - *e*-phonon scatting (  $\sim T^5$  low *T*,  $\sim T$  high *T*)
- More than one scattering mechanism may be operative; one has an effective \(\tau\) (given by the Matthiesen's rule)

$$\frac{1}{\tau} = \sum_{i} \frac{1}{\tau_i}$$

<u>Explains universal behaviour of good metals! So what</u>
VBS Fermions in Disorder – 22



### **Experiments and Puzzles**

- **9** Numbers: Typical metals  $ho \sim 10^{-6(-8)}$  Ohm-cm(m)
- Data by Mooij (1973),  $Ti_{1-x}AI_x$  alloys
  - Image of the second state is a second state of the second stat

• For large x, 
$$\frac{\mathrm{d}\rho}{\mathrm{d}T}$$
 is *negative*!!!

- All the resistivities are tending to a roughly equal saturation value!
- The saturation resisitivity 2 orders of magnitude higher than usual metallic values...



- Note that these are binary *alloys*...crystals with a random placing of Ti and Al ions! Electrons see a
- VBS **"highly disorderd" potential!**

Fermions in Disorder – 23



### There's more!

**•** Mooij found that the low temperature  $\frac{1}{\rho} \frac{d\rho}{dT} \equiv \alpha$  is

related to the "residual resistivity"



- This *is* magic! The key resistivity is about 100  $\mu\Omega$ -cm! If low T resistivity exceeds this value, then strange things happen...
- Most interestingly, similar stuff is seen in other disordered alloys!
  VBS
  Fermions in





### How can we forget the Cuprates?

**P** Resistivity in high  $T_c$  normal state



What is (are?) the puzzle(s) here?



# Story so far...

- The Bloch-Boltzmann theory is highly successful in explaining resistivities of elemental metals containing a small concentration of impurities
- **Experiments on disordered alloys suggest** 
  - Possibility of negative  $\frac{1}{\rho} \frac{d\rho}{dT}$
  - Sorrelated with low temperature resistivity; if low T resistivity  $\gtrsim 100 \ \mu\Omega$ -cm, we have negative  $\frac{1}{\rho} \frac{d\rho}{dT}$

- How do we understand this?
- Before we get to the answer, we need to understand resistivity and its relation to other response functions

#### **Meaning of Standard Formula for Conductivity**

The "standard result" for conductivity

$$\sigma = \frac{ne^2\tau}{m}$$

(all symbols have usual meanings)

- Meaning of  $\tau$  sharper meaning for  $\frac{1}{\tau} \frac{1}{\tau}$  is the number of collisions undergone by the electron per unit time...
- Idea: Think of an electron to be in a k state at time t = 0, then  $\tau$  is the life-time of such a state
- Also, the electron does not remember past collisions!



# Meaning of $\tau$

- To see that  $\tau$  is the life-time, ask what is the probability P(t) that the electron is still in the state k for  $t \leq \tau$ ...
- **•** The probability that the electron did not undergo a collision in an infinitesimal time  $\epsilon$  is  $\left(1 \frac{\epsilon}{\tau}\right)$ ...Thus,

$$P(t) = \lim_{N \to \infty} \left( 1 - \frac{1}{\tau} \frac{t}{N} \right)^N = e^{-t/\tau}$$

• This precisely connects up with our earlier statement: An electron placed in a Bloch state k evolves according to  $\psi(t) \sim \psi_k e^{-i\varepsilon(k)t - \frac{t}{2\tau_k}}$ ; "lifetime" is  $\tau_k \sim \tau$ ! Caveat: Note however that  $\tau$  is Drude formula is the transportation



## So what is the electron doing?

- A collision puts the electron in a different Bloch state (of same energy, discussion restricted to *impurity scattering*)...classically, simply changed direction...
- A moment's reflection tells us that the electron is RANDOM WALKING



Since the average velocity of electrons is  $v_F$ , the mean free path of electrons is  $\ell = v_F \tau$  Caveat: Note that this, in general, is not the "average spacing between impurities"

#### So what is the electron doing in an E-field?

**•** The electron is random walking and *drifting*!



**Develops a** drift velocity  $|\boldsymbol{v}_d| \sim \tau |\boldsymbol{E}|$ 



#### **Consequences of Random Walk**

- So what? How is this different from a *free* gas (no scattering) where electrons propagate with only Pauli to respect?
- Key question: Suppose we create a very long wavelength (compared to inverse Fermi vector) density disturbance (without changing the total number of electrons) in the electron gas...what difference does the scattering (dirt) make to this
- In the free gas, there is no mechanism to "relax" this density wave...and the system will simply "do some dynamics" ...can never attain a uniform density back again! We will call this the Free Fermi Fixed Point (more later...)!



#### **Consequences of Random Walk**

- What happens in a dirty gas?
- Particles, doing random walk, will DIFFUSE! Well known elementary result!
- Punch line: Presence of weak disorder will give us a qualitatively new state... a state with a diffusive density mode (not present in the free)...this state is the Diffusive Fixed Point
- Note that there will be no qualitative differences in the thermodynamic properties of DFP and FFFP!
- Why the word weak?
- What is "not weak" is the question that we will address in great detail...

#### **Consequences of Random Walk: Diffusion**

- At the diffusive fixed point (DFP), we are guaranteed that density fluctuations n(r, t) will be governed by
  - Continuity equation (conservation law) always holds (J particle current)

$$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

 Diffusive constitutive (Fick's) law (property of DFP)

$$\boldsymbol{J} = -D\boldsymbol{\nabla}n$$

*D* is the diffusion coefficient...property of the DFP Caveat: A more "correct" form is  $J(q, \omega) = -D(q, \omega)iqn(q, \omega)$ 

What determines D?



### The Diffusion Coefficient

- Consider dicing up space into cubes of size  $\ell^d$  (in d dimensions)
- Take three adjacent "cubes" (in 1-d) called -1, 0, 1 with  $N_{-1}$ ,  $N_0$  and  $N_1$  particles at time t = 0



• At time  $t = \tau$ , the number of particles in the 0 cube is  $\frac{1}{2}(N_{-1} + N_{+1})$ 



#### The Diffusion Coefficient

**•** Change in particle number  $\Delta N_0 = \frac{1}{2}(N_{-1} + N_{+1} - 2N_0)$ 



If we do this correctly in 3D, we will get

$$D = \frac{1}{3}v_F^2\tau$$

- We see that D is linearly related to  $\tau$ ...this is not the first time this has happened! Note that the conductivity is also linear in  $\tau$ !!!
- **•** Are  $\sigma$  and D related?



#### **•** Here is an experiment



- The "undisturbed" metal has electron density n and chemical potential  $\mu$
- **•** The battery generates an electric potential  $\phi(x)$
- Open circuit...there is not current flowing through the circuit!



### **Diffusion Coefficient and Conductivity**

- Since there is a potential gardient, there is obviously an electric current  $j = -\sigma \partial_x \phi$ ...it is *not* zero! Something fishy?
- No...there is another contribution to the current coming from diffusion...
- The chemical potential varies in space  $\mu(x) = \mu + e\phi(x)$  resulting is a density variation given by

$$n(x) \approx n + \frac{\partial n}{\partial \mu} e\phi$$

■ Diffusive particle current  $J = -D\frac{\partial n}{\partial \mu}e\partial_x\phi$ ...which contributes to a diffusive electric current  $j_D = -eJ = D\frac{\partial n}{\partial \mu}e^2\partial_x\phi$ 

#### Diffusion Coefficient and Conductivity

- **•** Now the total current must vanish  $j + j_D = 0$
- We get  $(g(\mu) \text{density of states at the chemical potential})$

$$\sigma = e^2 \frac{\partial n}{\partial \mu} D = e^2 g(\mu) D$$

This is the famed *Einstein relation*...diffusion and electrical conduction are *closely* related!

- This is a result of particle number conservation...and hence applicable in any system!!
- At the DFP, finite diffusion coefficient implies a finite conductivity!
- Our next step is to derive this relation from a formal
- VBS point of view...



## And Einstein Appears!

Solution We immediately see that the DC conductivity  $\sigma$  is realted to the diffusion constant D via

$$\sigma = e^2 g(\mu) D$$

- This is a rather general feature of the DFP...transport properties can all be finally related to *D* Exercise: How about thermal conductivity?..can you see Widemann-Franz?
- This is really a consequence of the underlying conservation laws...
- Solution When will  $\sigma$  go to zero? The case interesting for us is when D vanishes....i. e., "absence of diffusion" which takes us back to 1958!
- Punch line: When disorder is "not weak", i. e., "strong enough"...*D* will vanish and we get an *Anderson insulator*...i. e.,  $g(\mu) \neq 0$ , but is not an electrical conductor!



### ...followed by Anderson!

YSICAL REVIEW

VOLUME 109, NUMBER 5

MARCH 1, 195

#### Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON Bell Telephone Laboratories, Murray Hill, New Jersey (Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

#### This is the beginning of the field...



# Anderson Model (AM)

- This is the model introduced by Anderson in 1958
- Electrons move on a lattice
- At each site there is a random on-site potential  $w_i$ which is usually taken to be uniformly distributed between -W and W

$$-t\sum_{ij}\left(c_i^{\dagger}c_j+\mathbf{h.~c.}\right)+\sum_i\left(w_i-\mu\right)n_i$$

**•** Parameters :  $\mu/t$  and W/t

We shall use Edwards and Anderson models to gain an understanding of the disorder problem

#### **Closer Look at the Drudé Formula**

The Drude formula in d spatial dimensions

$$\sigma = \frac{ne^2\tau}{m} = \underbrace{\overbrace{(C_d k_F^d)}^n e^2 \underbrace{\ell}_{\ell}}_{\substack{m \underbrace{\left(\frac{\hbar k_F}{m}\right)}_{v_F}}} = \frac{e^2}{\hbar} C_d k_F^{d-2} \left(k_F \ell\right)$$

Question: What is  $\frac{e^2}{\hbar}$ ? (Ans: 0.25 milli- $\Omega^{-1}$ )

We will discuss d = 3 and d = 2 separately...but not d =

- Suddenly we realize what is "weak"...the disorder is "weak" when the mean free path is much larger than the inverse Fermi vector..., i. e., when  $k_F \ell \gg 1$ ...in this case then we have nothing to do...
- But what if k<sub>F</sub> l ~ 1 ??...can k<sub>F</sub> l ≪ 1???? If k<sub>F</sub> l ~ 1 the electron is moving only a distance of order of the inter-electron spacing (~ lattice spacing) between collisions...thus is essentially not moving!! There is trouble if the mean free path is less than the de Broglie wavelength of the electron!! Thus if the disorder is "not weak" the electron is unable to random walk...absence of diffusion!! k<sub>F</sub> l ~ 1 corresponds to the loffe-Regel limit!



### **loffe-Regel Conductivity in** d = 3

In d = 3 ( $a \sim$  lattice parameter,  $k_F = \frac{2\pi}{a}$ )

$$\sigma_{IR} = \frac{e^2}{\hbar} \frac{1}{3\pi^2} k_F \underbrace{(k_F \ell)}_{\sim 1} = \frac{e^2}{\hbar} \frac{2}{3\pi} \frac{1}{a}$$

The resistivity at such strong disorder is

$$\rho_{IR} = \underbrace{\frac{\hbar}{e^2}}_{4000\Omega} \underbrace{\frac{3\pi}{2}}_{4} \underbrace{\frac{3\pi}{2}}_{2.5\mathring{A}} \sim 400\mu\Omega\text{-cm}$$

this is definitely an over-estimate, but we are quite close to the low temperature resistivity of 100  $\mu\Omega$ -cm where trouble beings in the Mooij experiments...



#### **loffe-Regel Conductivity in** d = 3

- In the Edwards model, if  $\tau$  becomes small (either by a large  $n_i$  or by a large v (strength of the potential)), then we will get into the regime where  $k_F \ell \sim 1$
- We thus see why strong disorder causes trouble...the key point is that electron mean free path becomes too small for it to be able to diffuse!!
- Thus as the disorder is made stronger, the system goes to a "new phase" where there is no diffusion...we will call this the Anderson Fixed Point (AFP)...
- Question: What is the nature of the "phase transition" between these two "phases"...i.e., the "diffusive phase" and "Anderson phase"?
- Mott took Drudé seriously and concluded that metal must have a minimum conductivity...the famous Mott minimum!! Accoding to Mott the transition from the "diffusive phase" to the "Anderson phase" is "first order", i.e., accompanied by a discontinuous jump in the conductivity (This has turned out to be wrong, but it is a beautiful idea nevertheless...)
- In the remainder of these discussions we will understand the nature of this quantum phase transition...but before that lets see what 2d has in store for us..



### **loffe-Regel Conductivity in** d = 2

In d = 2 ( $a \sim$  lattice parameter,  $k_F = \frac{2\pi}{a}$ )

$$\sigma_{IR} = \frac{e^2}{\hbar} \frac{1}{\pi} k_F^0 \underbrace{(k_F \ell)}_{\sim 1} = \frac{e^2}{\hbar} \frac{2}{\pi}$$

The resistivity at such strong disorder is

$$\rho_{IR} = \underbrace{\frac{\hbar}{e^2}}_{4000\Omega} \frac{1}{2} \sim 20 \mathbf{k} \Omega$$

...this is amazing! The loffe-Regel resistivity turns out to be a universal number (independent of  $k_F$ ) in 2d...

Thus, if we see data that shows values of resistivity (or sheet resistance, as it is called in 2d), then we know we are in the strong disorder regime...



## Summary of Discussions

- Expect Drudé formula to hold for weak disorder
- "Weak" means  $k_F \ell \gg 1$
- Solution When  $k_F \ell \to 1$ , resistivity will increase...the typical order of magnitude when  $k_F \ell \to 1$  is called the loffe-Regel limit (~100  $\mu\Omega$ -cm (3d), ~10 K $\Omega$  (2d))...
- In any experimental system if we see low temperature resistivity greater than the IR limit it is suggestive
- As the strength of disorder increases, we will have a "new phase" where the mean free path is so small that it has no meaning, i. e., the electron will stop diffusing
- Key question: Is there a "phase transition" as a function of disorder strength? What is the nature of the transition if it exists?



## Some Preliminaries

- We recall some elementary things about states for further discussion
- The state  $|j\rangle$  represents the one particle state at site j
- The one particle states of the disordered Hamiltonian are denoted by  $|a\rangle$  (yes,  $|a\rangle$  depends on the realization of the disorder)
- Clearly

$$|a\rangle = \sum_{j} \langle a|j\rangle |j\rangle$$
, with  $\sum_{j} j|\langle a|j\rangle|^2 = 1$ 

By unitarity

$$|j\rangle = \sum \langle j|a\rangle \ |a\rangle$$







Extended states are "close to bloch states" and are "non zero" throughout the lattice



### **Extended States**

- **A electron in such an extended state can** *carry current*
- In such an extended state  $|a\rangle$ , very roughly

$$|\langle a|j\rangle| \sim \frac{1}{\sqrt{N}}$$

where N is the number of sites in the system... This is the statement of the idea the wavefunction is nonzero throughout the lattice Caveat: There are wave functions that satisfy this criterion, but *do not* carry currents...do you know any?





### **Localized States**

- **Associated length scale**  $\xi$  called the *localization* length
- **Does not carry a current**
- If we calculate the projection of such a state on to particular sites, we will find that there are some j for which

 $|\langle a|j\rangle| \sim 1$ 

...indeed these are those sites around which the state is localized!



# "Local" Questions

- How can we tell if a given state is extended or localized?
- What determines which states are extended and which states are localized?
- **Do we need a critical disorder to have localized states?**
- **\_** ...
- What has this got to do with diffusion?
- Our discussion (unless otherwise stated) will be valid only for 3d...2d will be discussed later in greater detail

#### Deciding Between Extended vs Localized

A popular method is to calculate the Inverse Participation Ratio (IPR) for the given state defined as

$$IPR(a) = \sum_{j} |\langle a|j \rangle|^4$$

- **•** For a Bloch state, IPR will turn out to be  $\frac{1}{N}$ , i. e., small for large N
- **•** For a state fully localized at one site say  $|i\rangle$ , IPR will be of the order unity
- Why the fourth power? There is a deeper reason...related to diffusion!!



# **Diffusion Again!**

- Ask the following question: Put an electron at site jat time t = 0, what is the probability  $P_j(T)$  of finding the electron at this same site at time T in the limit  $T \longrightarrow \infty$ ?
- **Solution** Ask a *second* question: Why ask the first question?
- Answer to the second question: If we find that  $P_j(T)$ goes to zero, then we know that the electron is random walking...more interestingly, if we find that  $P_j(T)$  is finite, then we know that the electron is *not* random walking!! If the second possibility is what we find, then we know that we will do not have diffusion!
- **•** How do we find  $P_j(T)$ ?



# **Diffusion Again!**

- Finding  $P_j(T)$  is quiet straightforward...let  $|\psi(t)\rangle$  be the state of the electron at time t
- Clearly,  $\psi(0) = |j\rangle = \sum_a \langle j | a \rangle | a \rangle$
- Now, since we "know" the one particle energy levels  $\epsilon_a$ , we have

$$|\psi(t)\rangle = \sum_{a} \langle j|a\rangle \, e^{-i\epsilon_a t} \, |a\rangle$$

**•** The probability that the electron is in |j
angle at time t is

$$P_{j}(t) = |\langle j|\psi(t)\rangle|^{2} = \left(\sum_{a} (\langle j|a\rangle)^{2} e^{-i\epsilon_{a}t}\right) \left(\sum_{b} (\langle b|j\rangle)^{2} e^{i\epsilon_{b}t}\right)$$
$$= \sum_{a} |\langle j|a\rangle|^{4} + \sum_{a\neq b} (\langle j|a\rangle)^{2} (\langle b|j\rangle)^{2} e^{-i(\epsilon_{a}-\epsilon_{b})t})$$



### **Diffusion Again!**

**Solution** To obtain  $P_j(T), (T \longrightarrow \infty)$ , one can argue that the *second term* does not contribute at large times, (Question: Argue this out! Suggestion: life can be made simple if you assume that  $\epsilon_a$  are non degenerate) and we obtain

$$P_j(T) = \sum_a |\langle j|a \rangle|^4$$

- Now assume that all states are extended...then we see immediately that  $P_j(T) \sim \frac{1}{N}$  and in the thermodynamic limit the particle diffuse away from  $|j\rangle$ !
- In the second scenario, assume that there is a localized state  $|\ell_j\rangle$  "centered" around *j*...then we know that  $|\langle \ell_j | j \rangle| \sim 1$ ...we thus see immediately that  $P_j(T)$  is finite and independent of N! This means that the particle is *not* diffusing!
- We also see the connection between the "fourth power" and diffusion!
- We have now answered our first question, of how to tell between localized and extended states...we now move on the next question...



### Which States are Localized?

- Consider an Anderson model with N lattice points in 3d
- First question: For a given W, are there any localized states?
- Second question: If there are localized states, "which" states are localized?
- The answer the first question for the Anderson model is: There are localized states for any finite W!
- ▲ moment's reflection will tell us that states very close to ε<sup>-</sup>, i.e., states deep in the Lifshitz tail are localized...in fact we used this fact to show that there are Lifshitz tails!
- Conjecture: states at the band centre are extended?



#### Which States are Localized?

- Will there be extended states in the Lifshitz tails? Are there localized states in the centre of the band? Is there "coexistance"?
- More generally, we can ask given an energy  $\varepsilon$ , what fraction of the total states  $g(\varepsilon)d\varepsilon$  are *localized*?
- Mott provided the answer : At a given energy  $\varepsilon$ , all states (in 3d) are either localized or extended! There is no "coexistance"!
- Mott's Argument: Suppose for a given realization of disorder, there is a localized state coexisting with extended states (all of which are infinitesimally close to energy  $\varepsilon$ )...now for another realization of disorder which is *"infinitesimally" different* from the one above, the localized states will hybridize with the extended Fermions in Disorder – 92

VBS states and become extended!!



# The Mobility Edge

• The arguments there exists an energy  $\mu_c^-$  which depends on the disorder W for which all states are localized... similarly, there is a  $\mu_c^+$ !



- The energy  $\mu_c^-(\mu_c^+)$  is called *mobility edge* Caveat: Mobility edges exist only in 3d
  - Natural question: How do  $\mu_c^{\pm}$  evolve with W?

### How Mobility Edge Evolves with Disorder

At large enough disorder we can "intuite" that the mobility edges will move towards the band centre



In fact, at a critical W<sub>c</sub>, all states become localized...it is this that was shown by Anderson in 1958



VBS

**2**dl

### **The** 3*d* **Disorder Solution!**

▲ At a given level of disorder  $W \le W_c$ , if the chemical potential  $\mu$  lies in the extended states, i.e.,  $\mu_c^- < \mu < \mu_c^- \text{ we have a diffusive metal, else an Anderson insulator!$ 





### **The** 1*d* **Disorder Solution!**

In 1d it turns out that W<sub>c</sub> is zero (Mott and Twose)! Any amount of disorder will localize all states! There is "not enough room" in 1d!



**2***d* is a bigger story! We will see that 2*d* is the *"marginal"* dimension!