System Identification

Lecture 2: Frequency domain methods: spectra, system responses, and estimated transfer functions

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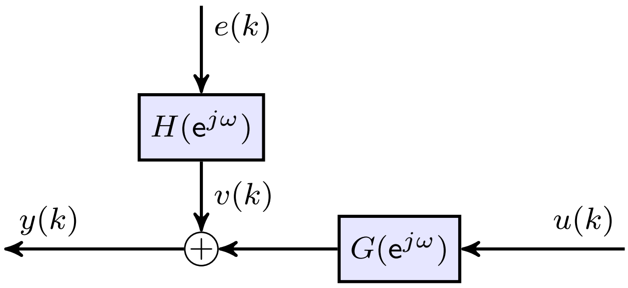
A simple ID method: sinusoidal correlation



Fundamental property of linear systems

$$Y(j\omega) = G(j\omega)U(j\omega) \quad \text{or, in discrete time,} \quad Y(e^{j\omega}) = G(e^{j\omega})U(e^{j\omega}).$$

Identification approach:

Apply a series of sinusoidal inputs ("sweptsine" input)

and for each find the gain and phase change of the output.

Sinusoidal correlation methods

Method:

Select the input frequency, ω_u (rad./sec.), such that,

$$\frac{\omega_u}{2\pi} = \frac{r}{NT} \quad \text{for some integer } r.$$

Input:

$$u(k) = \alpha \cos(\omega_u k), \quad k = 0, 1, \ldots, N-1$$

Output:

$$y(k) = \alpha \left| G(\mathrm{e}^{j\omega_u}) \right| \cos(\omega_u k + \theta(\omega_u)) + v(k) + \text{transient}$$

$$\text{where} \quad \theta(\omega_u) = \arg(G(\mathrm{e}^{j\omega_u}))$$

Sinusoidal correlation methods

Correlation functions:

$$I_c(N) = \frac{1}{N} \sum_{k=0}^{N-1} y(k) \cos(\omega_u k)$$

$$I_s(N) = \frac{1}{N} \sum_{k=0}^{N-1} y(k) \sin(\omega_u k)$$

These can be calculated from the data.

Expanding:

$$I_c(N) = \frac{\alpha}{2} \left| G(\mathrm{e}^{j\omega_u}) \right| \cos(\theta(\omega_u)) + \alpha \left| G(\mathrm{e}^{j\omega_u}) \right| \frac{1}{N} \sum_{k=0}^{N-1} \cos(2\omega_u k + \theta(\omega_u)) + \frac{1}{N} \sum_{k=0}^{N-1} v(k) \cos(\omega_u k)$$

Sinusoidal correlation methods

If the noise, $v(k)$, is sufficiently uncorrelated

(for example, is filtered gaussian noise),

then the variance satisfies,

$$\lim_{N \longrightarrow \infty} \text{var} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} v(k) \cos(\omega_u k) \right\} \; = \; 0$$

with a convergence rate of $1/N$.

Sinusoidal correlation methods

So, as in the limit as $N \longrightarrow \infty$,

$$E\{I_c(N)\} \longrightarrow \frac{\alpha}{2} \left| G(\mathrm{e}^{j\omega_u}) \right| \cos(\theta(\omega_u))$$

$$E\{I_s(N)\} \longrightarrow \frac{-\alpha}{2} \left| G(\mathrm{e}^{j\omega_u}) \right| \sin(\theta(\omega_u))$$

and $\lim_{N \longrightarrow \infty} \text{var}\{I_c(N)\} = 0, \quad \lim_{N \longrightarrow \infty} \text{var}\{I_s(N)\} = 0$

Estimate the transfer function via:

$$\left| \hat{G}_N(\mathrm{e}^{j\omega_u}) \right| = \frac{\sqrt{I_c(N)^2 + I_s(N)^2}}{\alpha/2}$$

$$\arg \hat{G}_N(\mathrm{e}^{j\omega_u}) = -\arctan \frac{I_s(N)}{I_c(N)}$$

Sweptsine ID methods

Advantages:

- Energy is concentrated at the frequencies of interest.
- Size of $u(k)$ can easily be tuned as a function of frequency.
- Easy to avoid saturation and tune S/N ratio.

Disadvantages:

- A large amount of data is required.
- Significant amount of time required for experiments.
- Some processes won't allow sinusoidal inputs.

Background on discrete signals and signal analysis

Signals considered in detail:

- Finite energy;
- Periodic;
- Random;
- Finite length.

Signal properties that we can estimate from data:

- Autocorrelation
- Crosscorrelation
- Frequency domain representation (via Fourier transform)
- Spectral densities (energy or power)

Fourier transform

Discrete-time domain signal:

$$x(k), \quad k = -\infty, \ldots, \infty.$$

The Fourier Transform of $x(k)$ defined as,

$$X(\mathrm{e}^{j\omega}) \;=\; \sum_{k=-\infty}^{\infty} x(k)\mathrm{e}^{-j\omega k} \quad \text{(or sometimes } X(\omega))$$

$X(\mathrm{e}^{j\omega})$ is periodic, with period 2π.

If $x(k)$ is finitely summable,

$$\sum_{k=-\infty}^{\infty} |x(k)| \;<\; \infty,$$

then $X(\mathrm{e}^{j\omega})$ converges.

Inverse Fourier transform

The inverse Fourier Transform is,

$$x(k) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega k} d\omega,$$

where $\quad k = -\infty, \ldots, \infty.$

Energy spectral density

If $x(k)$ is a finite energy signal,

$$\|x(k)\|_2^2 = \sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty.$$

The sequence, $x(k)$, has a Fourier transform,

$$X(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)\mathrm{e}^{-j\omega_k}, \quad \text{where} \quad \omega \in [-\pi, \pi),$$

The *energy* spectral density can be defined as,

$$S_x(\mathrm{e}^{j\omega}) = |X(\mathrm{e}^{j\omega})|^2.$$

Autocorrelation (finite energy signals)

The autocorrelation of $x(k)$ is,

$$R_x(\tau) = \sum_{k=-\infty}^{\infty} x(k)x(k-\tau), \tau = -\infty, \ldots, 0, \ldots, \infty.$$

The energy spectral density is the Fourier Transform of the autocorrelation:

$$\sum_{\tau=-\infty}^{\infty} R_x(\tau)\mathrm{e}^{-j\omega\tau} = S_x(\mathrm{e}^{j\omega}).$$

Discrete periodic signals

If $x(k)$ is periodic with period equal to N (assume N even);

$$x(k) \;=\; x(k+N), \quad \text{for all } k \in \{-\infty, \infty\}.$$

The fundamental frequency is,

$$\omega_0 \;=\; \frac{2\pi}{N}.$$

There are only N unique harmonics of the sinusoid, $\mathrm{e}^{j\omega_0}$.

The non-negative harmonic frequencies are,

$$\mathrm{e}^{jn\omega_0}, \quad n = 0, 1, \ldots, N/2.$$

Discrete Fourier series (periodic signals)

Periodic signal: $x(k)$ (period $= N$).

The Fourier series is:

$$X(\mathrm{e}^{j\omega_n}) \;=\; \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega_n k}, \quad \text{where} \quad \omega_n = \frac{2\pi n}{N} = n\omega_0,$$

$$n = 0, \ldots, N-1.$$

The inverse transform is:

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(\mathrm{e}^{j\omega_n})\mathrm{e}^{j\omega_n k}.$$

Autocorrelation (periodic signals)

The autocorrelation of $x(k)$ (of period N) is:

$$R_x(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)x(k-\tau).$$

The Fourier transform of $R_x(\tau)$ is defined as the power spectral density,

$$\phi_x(\mathrm{e}^{j\omega_n}) \;=\; \sum_{\tau=0}^{N-1} R_x(\tau)\mathrm{e}^{-j\omega_n\tau} \;=\; \frac{1}{N}|X(\mathrm{e}^{j\omega_n})|^2$$

Energy in a single period:

$$\sum_{k=0}^{N-1}|x(k)|^2 = \sum_{n=0}^{N-1}\phi_x(\mathrm{e}^{j\omega_n})$$

Cross-correlation (periodic signals)

The cross-correlation of $y(k)$ and $u(k)$ (both of period $= N$) is:

$$R_{yu}(\tau) = \frac{1}{N}\sum_{k=0}^{N-1} y(k)u(k-\tau).$$

Cross-spectral density (FT of the cross-correlation):

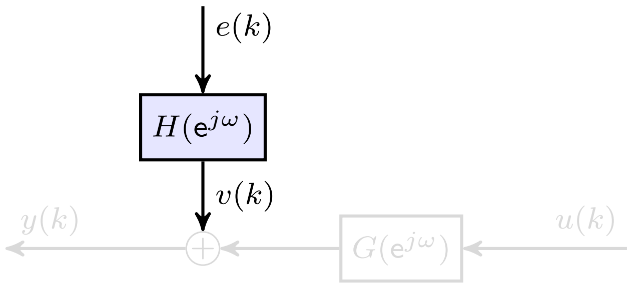
$$\begin{aligned}\phi_{yu}(\mathrm{e}^{j\omega_n}) &= \sum_{\tau=0}^{N-1} R_{yu}(\tau)\mathrm{e}^{-j\omega_n\tau}, \qquad \omega_n = \tfrac{2\pi n}{N}, \\ & \qquad\qquad\qquad\qquad\qquad\quad n = 0,\ldots,N-1 \\ &= \frac{1}{N}Y(\mathrm{e}^{j\omega_n})U^*(\mathrm{e}^{j\omega_n})\end{aligned}$$

Noise models: random signals

Normally distributed noise:

$$e(k) \in \mathcal{N}(0, \lambda), \implies \begin{cases} E\{e(k)\} = 0 & \text{(zero mean)} \\ E\{|e(k)|^2\} = \lambda & \text{(variance)} \end{cases}$$

The $e(k)$ are independent and identically distributed (i.i.d.).



$$v(k) \;=\; \sum_{l=0}^{\infty} h(l)e(k-l) \;=\; H\,e(k) \quad \text{with } e(k) \in \mathcal{N}(0, \lambda).$$

Autocovariance (random signals)

For random $x(k)$, with $E\{x(k)\} = 0$,

Define the autocovariance sequence,
or covariance function, as;

$$\begin{aligned} R_x(\tau) &= E\{x(k)x(k-\tau)\} \\ &= E\{x(k)x^*(k-\tau)\} \quad \text{(in the complex case)} \\ &= E\{x(k)x^*(k-\tau)\} \quad \text{(in the multivariable case)} \end{aligned}$$

General (non-stationary, non-zero mean) case:

$$\begin{aligned} R_x(s,t) \;&=\; E\{(x(s) - E\{x\})(x(t) - E\{x\})\} \\ &= E\{x(s)x(t)\} \quad \text{(if zero mean)} \\ &= R_x(s-t) \quad \text{(if stationary)} \end{aligned}$$

Power spectral density (random signals)

The *power* spectral density is defined as the Fourier transform of $R_x(\tau)$,

$$\phi_x(\mathrm{e}^{j\omega}) := \sum_{\tau=-\infty}^{\infty} R_x(\tau)\mathrm{e}^{-j\omega\tau} \quad \text{where} \quad \omega \in [-\pi, \pi).$$

The inverse transform is given by,

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_x(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega\tau} d\omega.$$

For a zero-mean random signal,

$$\lim_{N\longrightarrow\infty} \frac{1}{N}\sum_{k=0}^{N-1} |x(k)|^2 = \mathrm{var}\{x(k)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_x(\mathrm{e}^{j\omega})\, d\omega$$

Basic properties

Autocovariance:

$$R_x(-\tau) = R_x^*(\tau)$$
$$R_x(0) \geq |R_x(\tau)| \quad \text{for all } \tau > 0$$

Spectral density:

$$\phi_x(\mathrm{e}^{j\omega}) \in \mathcal{R}$$
$$\phi_x(\mathrm{e}^{j\omega}) \geq 0 \quad \text{for all } \omega$$
$$\phi_x(\mathrm{e}^{j\omega}) = \phi_x(\mathrm{e}^{-j\omega}) \quad \text{for all real-valued } x(k)$$

Cross-covariance (random signals)

For random $y(k)$ and $u(k)$, the cross-covariance is:

$$R_{yu}(\tau) = E\{(y(k) - E\{y(k)\})(u(k-\tau) - E\{u(k)\})\}$$

For zero mean signals, $E\{y(k)\} = 0$ and $E\{u(k)\} = 0$,

$$R_{yu}(\tau) = E\{y(k)u(k-\tau)\}$$

Joint stationarity is required to make the definition dependent on τ alone.

If $R_{yu}(\tau) = 0$ for all τ then $y(k)$ and $u(k)$ are *uncorrelated*.

Cross power spectral density (random signals)

The Fourier transform of $R_{yu}(\tau)$ is defined as the cross spectral density, or cross-spectrum,

$$\phi_{yu}(e^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau)e^{-j\omega\tau}, \quad \omega \in [-\pi, \pi).$$

The inverse is,

$$R_{yu}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{yu}(e^{j\omega})e^{j\omega\tau}\,d\omega.$$

Discrete-Fourier Transform (finite-length signals)

Finite length signal,

$$x(k), \quad k = 0, \ldots, N-1.$$

The Discrete Fourier Transform (DFT) of $x(k)$ is:

$$X(\mathrm{e}^{j\omega_n}) = \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-j\omega_n k}, \quad \text{where} \quad \omega_n = \frac{2\pi n}{N},$$

$$n = 0, \ldots, N-1.$$

The inverse DFT is,

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(\mathrm{e}^{j\omega_n})\mathrm{e}^{j\omega_n k}, \quad k = 0, \ldots, N-1.$$

Periodogram

The periodogram (for a random signal $v(k)$) is defined as:

$$\frac{1}{N} \left| V_N(\mathrm{e}^{j\omega}) \right|^2$$

See [Schuster, 1900] for an interesting application.

Asymptotically unbiased estimator of the spectrum:

$$\lim_{N \longrightarrow \infty} E \left\{ \frac{1}{N} |V_N(\mathrm{e}^{j\omega})|^2 \right\} = \phi_v(\omega)$$

This assumes that the autocorrelation decays quickly enough:

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{\tau=-N}^{N} |\tau R_v(\tau)| = 0$$

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