System Identification

Lecture 12: ARX, Instrumental variables, Validation

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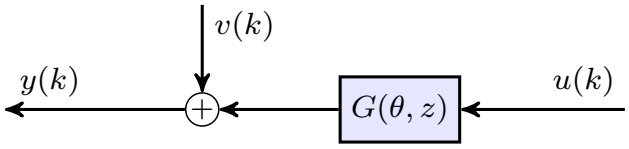
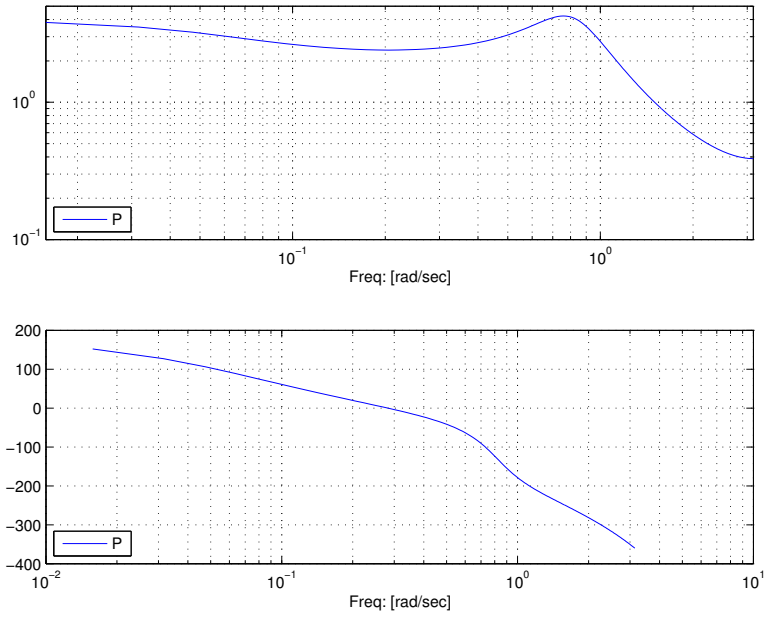
ARX estimation

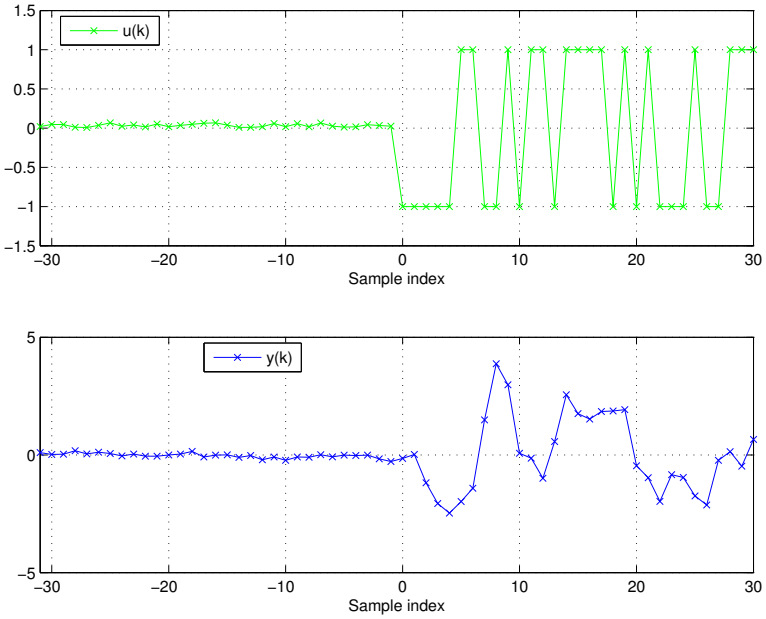


$$G(z) = \frac{B(\theta, z)}{A(\theta, z)}$$



ARX estimation

Experiment: $N = 31$



$$v(k) \sim \mathcal{N}(0, \lambda)$$

$$\lambda = 0.005$$

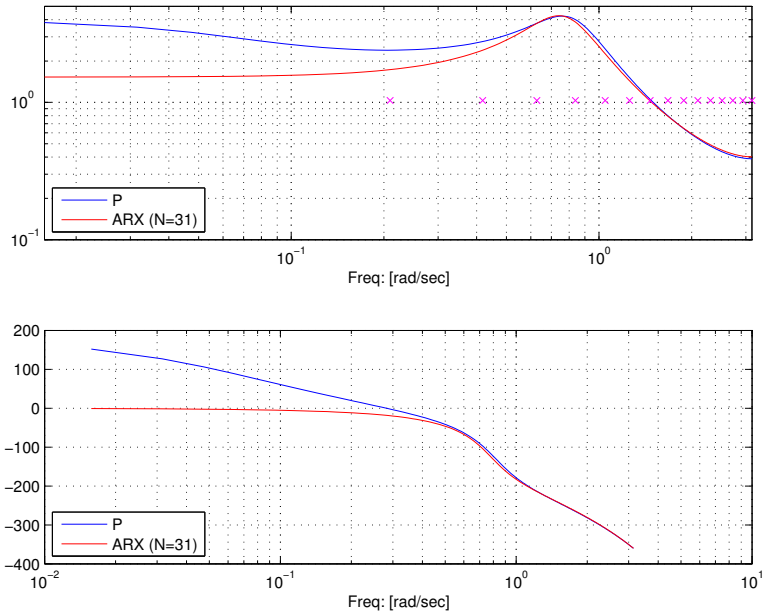
ARX example: MATLAB calculations

```
um = ...                                        % input measurements
ym = ...                                        % output measurements

for k = 0:N-1,
    vphi = [-ym(k-1) -ym(k-2) -ym(k-3) um(k-2) um(k-3)]';
    Rexpt = Rexpt + (1/N)*vphi*vphi';
    fexpt = fexpt + (1/N)*vphi*ym(k);
end

theta = Rexpt\fexpt;                            % least-squares fit
Bvec = [0; 0; theta(4); theta(5)];
Avec = [1; theta(1); theta(2); theta(3)];

Ghat = tf(Bvec,Avec,-1);
```

For clarity the negative time initializations and negative index details are not shown.

ARX estimation

Estimate: $N = 31$

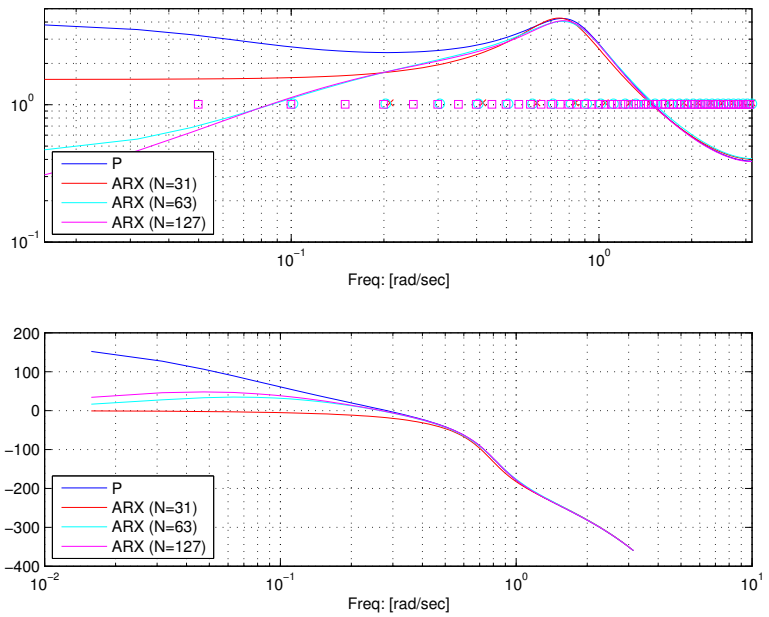


$A(\theta, z)$	plant	est.
a_1	-2.08	-1.81
a_2	1.71	1.37
a_3	-0.61	-0.39

$B(\theta, z)$	plant	est.
b_1	1.00	1.02
b_2	-1.10	-0.86

ARX estimation

Estimate: $N = 127$

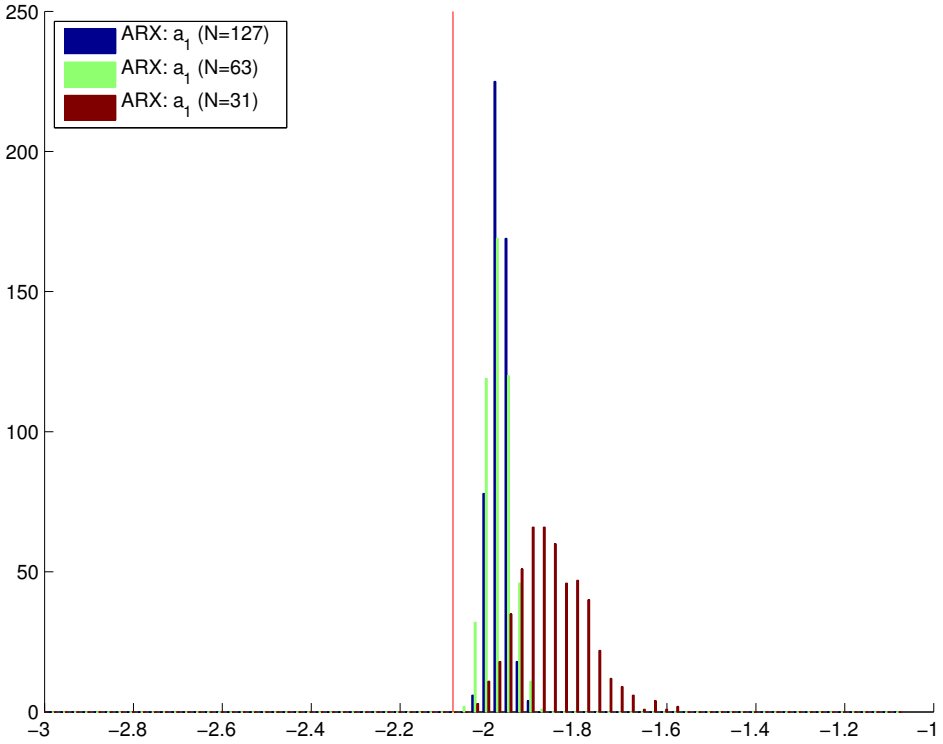


$A(\theta, z)$	plant	est.
a_1	-2.08	-2.00
a_2	1.71	1.62
a_3	-0.61	-0.55

$B(\theta, z)$	plant	est.
b_1	1.00	1.04
b_2	-1.10	-1.05

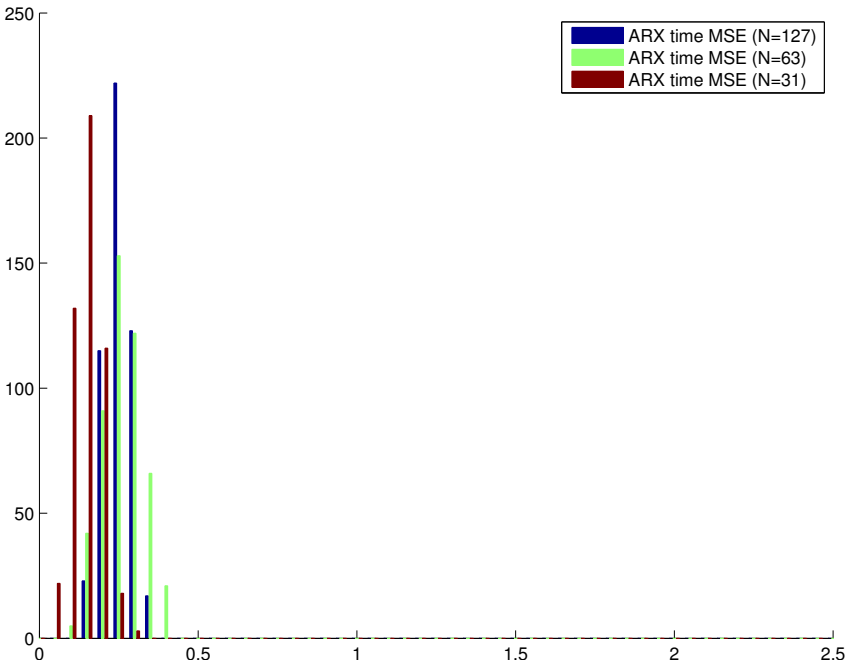
ARX estimation

Statistics for 500 ARX estimation experiments



ARX estimation

Time-domain MSE for 500 ARX estimation experiments



Aside: Wilkinson's polynomial

Nominal polynomial

$$w(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)\dots(x-20).$$

The roots of $w(x)$ are $\{1, 2, 3, \cdots, 20\}$.

Expanding gives,

$$\begin{aligned} w(x) = x^{20} &- 210x^{19} + 20615x^{18} - 1256850x^{17} + 53327946x^{16} - 1672280820x^{15} \\ &+ 40171771630x^{14} - 756111184500x^{13} + 11310276995381x^{12} \\ &- 135585182899530x^{11} + 1307535010540395x^{10} \\ &- 10142299865511450x^{9} + 63030812099294896x^{8} \\ &- 311333643161390640x^{7} + 1206647803780373360x^{6} \\ &- 3599979517947607200x^{5} + 8037811822645051776x^{4} \\ &- 12870931245150988800x^{3} + 13803759753640704000x^{2} \\ &- 8752948036761600000x + 2432902008176640000. \end{aligned}$$

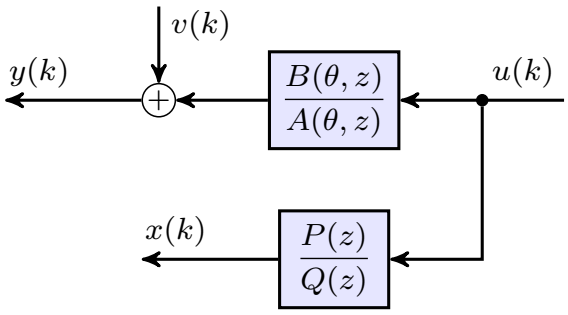
Aside: Wilkinson's polynomial

Perturbed polynomial

Modify the x^{19} coefficent: -210 is replaced by -210.0000001192

Nominal	Perturbed
1	1.00000
2	2.00000
3	3.00000
4	4.00000
5	5.00000
6	6.00001
7	6.99970
8	8.00727
9	8.91725
10, 11	$10.09527 \pm 0.64350i$
12, 13	$11.79363 \pm 1.65233i$
14, 15	$13.99236 \pm 2.51883i$
16, 17	$16.73074 \pm 2.81262i$
18, 19	$19.50244 \pm 1.94033i$
20	20.84691

Instrumental variable estimation



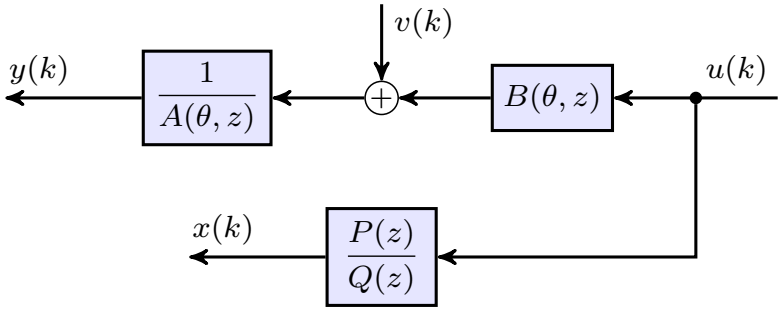
$$v(k) \sim \mathcal{N}(0, \lambda)$$

$$\lambda = 0.005$$

Approach:

1. Estimate $\hat{\theta}_{\mathsf{LS}}$ via linear regression.
2. Select $Q(z) = \hat{A}_{\mathsf{LS}}(z)$ and $P(z) = \hat{B}_{\mathsf{LS}}(z)$.
3. Calculate $\hat{\theta}_{\mathsf{IV}}$.

Instrumental variable estimation

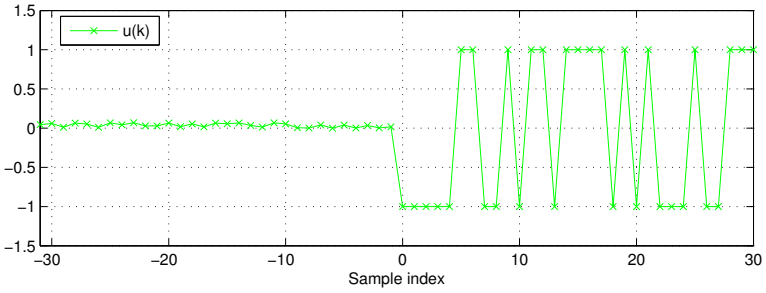


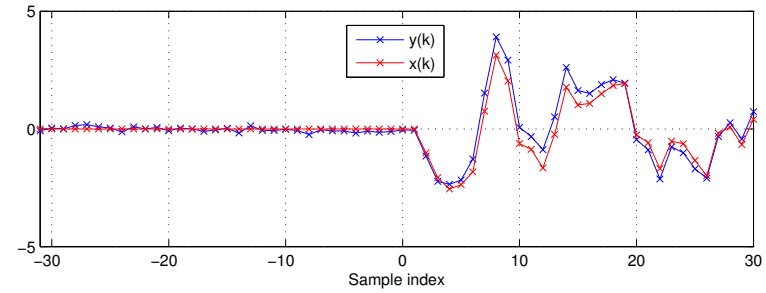
$$\zeta(k) = \begin{bmatrix} -x(k-1) & \dots & -x(k-n) & u(k-1) & \dots & u(k-m) \end{bmatrix}$$

$$\hat{\theta}_{\mathsf{IV}} = \left(\frac{1}{N} \sum_{k=0}^{N-1} \zeta(k) \varphi^T(k) \right)^{-1} \frac{1}{N} \sum_{k=0}^{N-1} \zeta(k) y(k).$$

Instrumental variable estimation

Experiment: $N = 31$



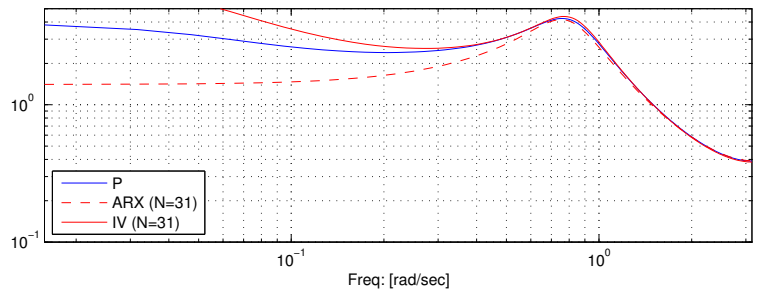


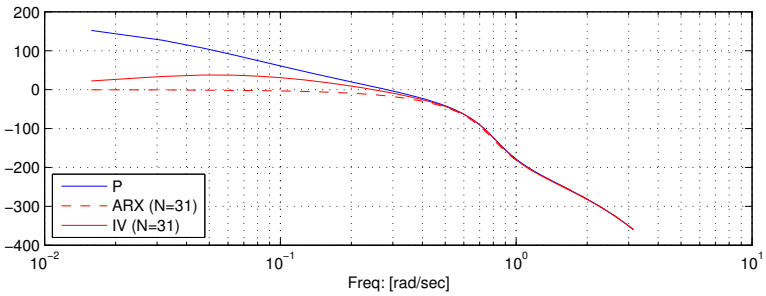
$v(k) \sim \mathcal{N}(0, \lambda)$

$\lambda = 0.005$

Instrumental variable estimation

Estimate: $N = 31$



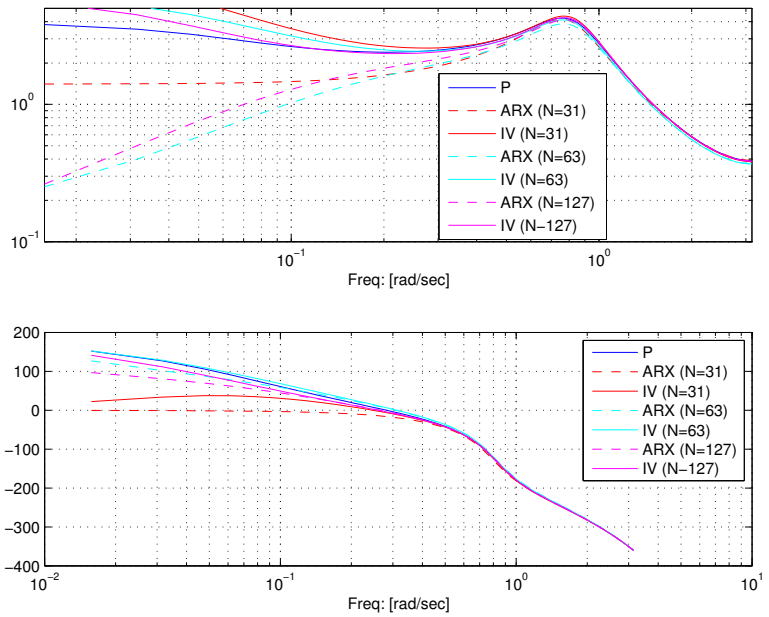


$A(\theta, z)$	plant	est.
a_1	-2.08	-2.16
a_2	1.71	1.82
a_3	-0.61	-0.67

$B(\theta, z)$	plant	est.
b_1	1.00	1.00
b_2	-1.10	-1.16

Instrumental variable estimation

Estimate: $N = 127$

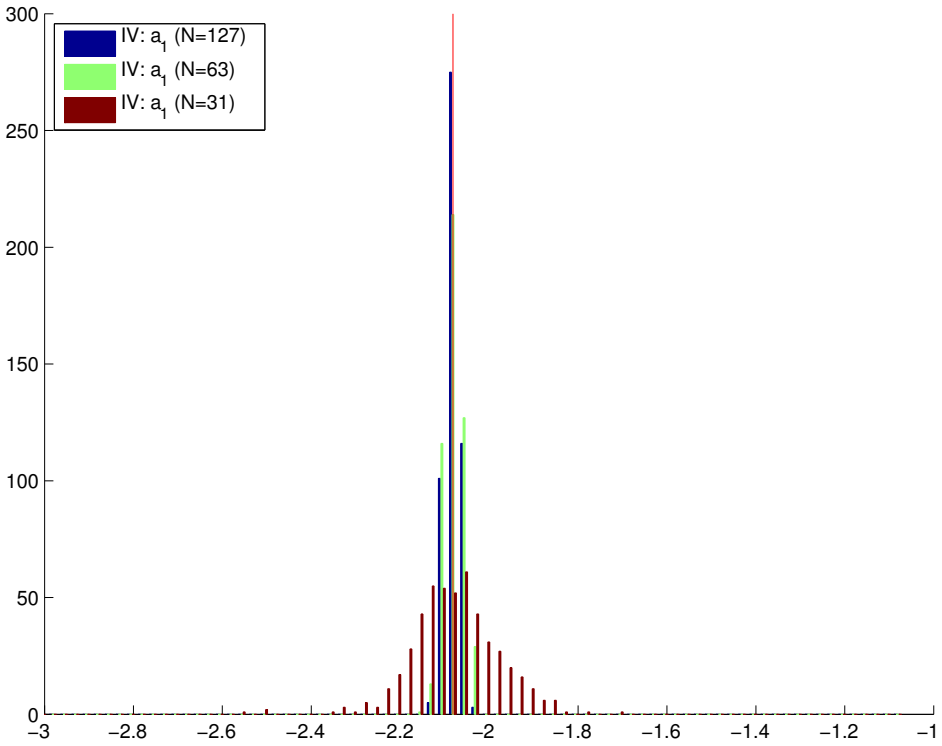


$A(\theta, z)$	plant	est.
a_1	-2.08	-2.10
a_2	1.71	1.74
a_3	-0.61	-0.63

$B(\theta, z)$	plant	est.
b_1	1.00	1.01
b_2	-1.10	-1.10

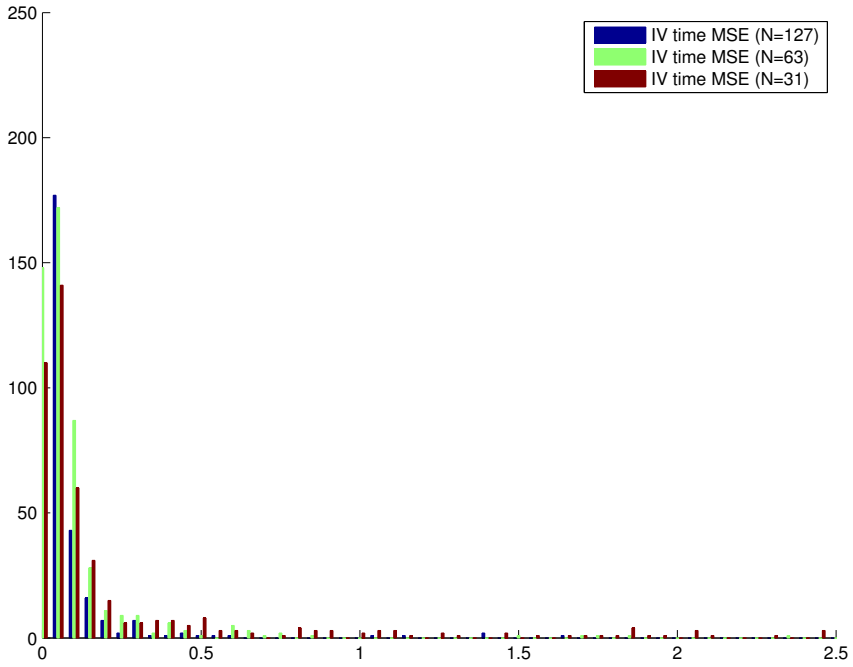
Instrumental variable estimation

Statistics for 500 ARX estimation experiments



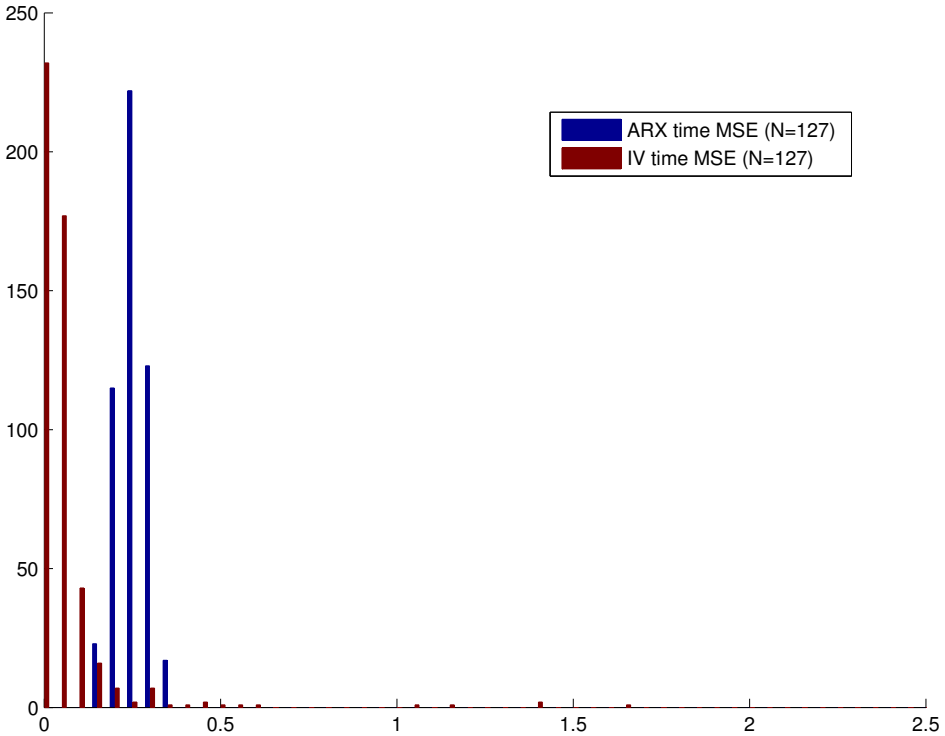
Instrumental variable estimation

Time-domain MSE for 500 ARX estimation experiments



ARX/IV mean-square error comparions

Time-domain MSE for 500 ARX estimation experiments



Validation

Model quality assessment

- Identification experiments, $Z_{\text{expt}} = \{y(k), u(k)\}, \quad k = 0, \ldots, N-1.$
- Model fitting:
  - Least-squares (ARX);
  - Pseudo-linear regressions;
  - Instrumental variable methods;
  - Nonlinear optimization.
- Validation experiments, $Z_{\text{val}} = \{y(k), u(k)\}, \quad k = 0, \ldots, N_{\text{val}}-1.$
- Compare models on validation data:
  - Mean-square error;
  - Residual characteristics.
- Select "best" model.

Identification approach

Typical workflow

- Assess basic characteristics, signal levels, noise, drift, saturation, etc.
- Build a "typical" simulation model for your system.
- Develop experimental procedures (on the simulation system):
  - Data lengths;
  - Input signal design.
- Test identification algorithms' capabilities (on the simulation system).
- Run experiments.
- Fit a selection of models (various structures).
- Run validation experiments.
- Assess models and select the best model.

Bibliography

Prediction error minimization

Lennart Ljung, *System Identification;Theory for the User,* 2nd Ed., Prentice-Hall, 1999, [sections 7.1, 7.2 & 7.3].

Parameter estimation statistics

Lennart Ljung, *System Identification;Theory for the User,* 2nd Ed., Prentice-Hall, 1999, [section 7.4].

Correlation and instrumental variable methods

Lennart Ljung, *System Identification;Theory for the User,* 2nd Ed., Prentice-Hall, 1999, [sections 7.5 & 7.6].