LECTURE 1

Naive Riemannian geometry. Topological manifolds.

(1) The statement "a metric is a matrix-valued function" is misleading, and we'll improve this definition. The way you can think of an inner product as a matrix is by first choosing a basis for your vector space. If e_i is your basis, then any vector v can be written uniquely as $v = \sum v^i e_i$ where v^i are scalars. Given an inner product, one recovers a matrix by setting $g_{ij} := \langle e_i, e_j \rangle$. Conversely, given a symmetric non-degenerate matrix g, one recovers an inner product by setting

$$\langle u, v \rangle := u^T g v$$

where u^T is the transpose of u. It is a so-called row vector, whose entries are given by the u^i, while v is a column vector.

(2) As I emphasized, a Riemannian metric is a choice of inner product at each tangent space. The "matrix" definition is most concrete, but obviously requires a choice of basis for each tangent space, which we have taken to be the "standard basis," whatever you think that means.

(3) Immersions need not be injective. Examples included two disjoint points being sent to a single point, and a real line wrapping around a circle over and over again.

(4) The reason for the square root? You'll see in homework the determinant inherits a factor of $\det(df_x)^2$ when you have an isometry f. So for volume to remain invariant, you'll want to take the square root of $\det g$.