LECTURE 10

Algebraic formulas

I know it's easy to get lost in everything, so let me summarize our algebraic tools.

1. Summary

1.1. The main players. There are two main algebraic players: First, we have the set of all deRham differential forms:

$$\Omega^\bullet_{deR}(M)$$

which is a ring. Its unit is the constant function $1 \in \Omega^0_{deR}(M)$. Its product is given by wedging together, pointwise.

The other player is

$$\Gamma(TM)$$

also known as vector fields. It is a Lie algebra.

1.2. Main properties. Skipping definitions, here are all the properties of the algebraic operations you need to know:

- d_{deR} is a degree 1 derivation $\Omega^\bullet_{deR}(M) \to \Omega^\bullet_{deR}(M)$. It squares to zero.
- For any $X \in \Gamma(TM)$, $i_X : \Omega^\bullet_{deR}(M) \to \Omega^\bullet_{deR}(M)$ is a degree *minus* 1 derivation. It squares to zero.
- For any X, $\mathcal{L}_X : \Omega^\bullet_{deR}(M) \to \Omega^\bullet_{deR}(M)$ is a degree zero derivation. It commues with d_{deR}.

1.3. Useful formulas. Finally, here are the algebraic formulas you need to know.

- Cartan's magic formula,

$$\mathcal{L}_X := d_{deR} \circ \iota_X + \iota_X \circ d_{deR}$$

- How to compute the directional derivative of the values of a differential form:

$$\mathcal{L}_X(\alpha(Y_1, \dots, Y_k)) = (\mathcal{L}_X\alpha)(Y_1, \dots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \dots, Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, \dots, Y_k).$$

- How the deRham derivative changes the values of a form:

$$\begin{aligned} d\alpha(Y_0, \ldots, Y_k) &= \sum_{i=0}^{k} (-1)^i Y_i(\alpha(Y_0, \ldots, \hat{Y}_i, \ldots, Y_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_k). \end{aligned}$$

That's it. Now I'll review the definitions of everything for your convenience.

2. The Lie derivative

Given any $X \in \Gamma(TM)$, one has maps

$$\mathcal{L}_X : \Gamma(TM) \to \Gamma(TM), \qquad \mathcal{L}_X : \Omega^k_{deR}(M) \to \Omega^k_{deR}(M).$$

which go by the same notation. To define this, we needed the theorem about ODEs: X defines a vector field, so you can flow along it. You evaluate α at that point. Since flowing is a diffeomorphism, you can pull back the value of α to the place you started before the flow.

$$(\Phi_t^X)^* \alpha(\Phi_t^X(x)) \in E_x.$$

Here, E is one of the vector bundles

$$TM, \qquad T^*M, \qquad \Lambda^k(T^*M),$$

and α is a section of E. The Lie derivative says let's measure the rate of change of this vector with respect to t:

$$\mathcal{L}_X(\alpha)(x) := \lim_{t \to 0} \frac{(\Phi_t^X)^* \alpha(\Phi_t^X(x)) - \alpha(x)}{t}$$

3. The Lie bracket

The Lie bracket was defined formally—if X and Y are two derivations from $C^\infty(M)$ to itself, we saw that while $X \circ Y$ may not be a derivation, the operation $X \circ Y - Y \circ X$ is. So we defined

$$[X, Y] := X \circ Y - Y \circ X \in Der(C^\infty(M), C^\infty(M)).$$

Miraculously, this purely algebraic definition agrees with a more geometric operation:

$$\mathcal{L}_X Y = [X, Y].$$

We proved this last class. One can also see that

$$\mathcal{L}_X f = df(X) = X(f).$$

Making sure you can prove this is a very good way to check that you understand all the basic definitions.

4. Differential forms eat tangent vectors

There is an isomorphism, for each k (and any finite-dimensional vector space V),

$$\Lambda^k(V^\vee) \to (\Lambda^k(V))^\vee$$

which sends an element of the form $\alpha_1 \wedge \ldots \wedge \alpha_k$ to the linear map

$$v_1 \wedge \ldots \wedge v_k \mapsto \det(\alpha_i(v_j)).$$

Since a differential form $\alpha \in \Omega^k(M)$ defines an element

$$\alpha(x) \in \Lambda^k((T^*M)_x)$$

for any $x \in M$, the isomorphism tells us we can think of $\alpha(x)$ as an element of

$$(\Lambda^k((TM)_x))^\vee$$

i.e., as something that eats k tangent vectors to x and spits out a number.

You can check that if $f : M' \to M$ is a smooth map, then

$$((f^*\alpha)(x))(v_1, \ldots, v_k) = \alpha(f(x))(Df_x(v_1), \ldots, Df_x(v_k)).$$

5. Contraction/interior multiplication

Given $X \in \Gamma(TM)$, we can define a map

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

by sending

$$\alpha \mapsto \alpha(X, -, \ldots, -).$$

This is defined using the isomorphism from the previous section. So for instance $\iota_X(\alpha)$ eats in $k-1$ vector fields $Y_1, \ldots, Y_{k-1}$ and spits out the function

$$\alpha(X, Y_1, \ldots, Y_{k-1}).$$

6. deRham derivative d_{deR}

I write both d_{deR} and d depending on the context. They are the same thing. For every k, there is a map

$$d^k : \Omega^k(M) \to \Omega^{k+1}(M)$$

and I lazily call each of these maps d.

In local coordinates, this operation was defined as follows. Any k-form α on $U \subset \mathbb{R}^n$ can be written

$$\alpha = \sum_I f_I dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where the index $I = (i_1 < \ldots < i_k)$ runs through every sequence of k increasing integers between 1 and n. And f_I is a choice of smooth function on U. Then

$$d\alpha := \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Though this is local, you can see that this defines a map of section $\Omega^k(M) \to \Omega^{k+1}(M)$ by checking that this definition is compatible with the transition functions for the bundles $\Lambda^k(T^*M)$ and $\Lambda^{k+1}(T^*M)$.

Remark 10.1. While formulas in local coordinates let you compute, those of you more formally-minded may ask for a more intrinsic characterization of the deRham derivative. There is a non-trivial theorem (see Conlon, Theorem 7.5.5 and its corollaries) asserting that

$$\Gamma(\Lambda^k(T^*M)) \cong \Lambda^k(\Gamma(T^*M)).$$

The lefthand side Λ takes exterior powers over $\mathbb{R}$; the righthand side takes exterior powers over $C^\infty(M)$. So by the righthand side, demanding that d be a derivation means we only need specify it on 0-forms (functions) and 1-forms. This is what we did in class. (In fact, there is another universal property that characterizes d simply by declaring it on 0-forms; this will be an optional problem on a homework.)

7. Some computational exercises

This is just a sanity check to make sure you know what you're doing. If you cannot do these, definitely come and talk to me and Phil. In what follows, everything is defined on $\mathbb{R}^3$.

- $\alpha = x^2 dx + y^2 dy + xydz$.
- $\beta = xyzdx \wedge dy + zdy \wedge dz$.
- $\omega = xdy \wedge dx + xdx \wedge dy$.
- $\eta = dx \wedge dy \wedge dz$.
- $X = x\frac{\partial}{\partial x}$.
- $Y = x^2 \frac{\partial}{\partial y}$.

(a) Show $\omega = 0$.
(b) Compute $\iota_X \alpha$.
(c) Compute $\iota_Y \beta$.
(d) Compute $\iota_Y \eta$.
(e) Compute $\alpha \wedge \beta$.
(f) Compute $\beta \wedge \alpha$.
(g) Compute $\mathcal{L}_X \alpha$.
(h) Compute $\mathcal{L}_Y \beta$.

(i) Compute $d\alpha$.
(j) Compute $d\beta$.
(k) Compute $[X, Y]$.

8. Solutions

(a) **Show $\omega = 0$.** Use that $dy \wedge dx = -dx \wedge dy$. More pedantically, at any point $p \in \mathbb{R}^3$, we have that

$$\omega(p) = x(p)dy|_p \wedge dx|_p + x(p)dx|_p \wedge dy|_p \in \Lambda^2((T^*\mathbb{R}^3)_p).$$

And this vector space was defined by quotienting out $(T^*\mathbb{R}^3)_p \otimes (T^*\mathbb{R}^3)_p$ by imposing the relation $v \otimes v = 0$, which means that $v \wedge w := [v \otimes w] = -[w \otimes v] =: -w \wedge v$. Long story short, we know that $dy|_p \wedge dx|_p = -dx|_p \wedge dy|_p$, so $\omega(p) = 0$ for all p. This means that ω is indeed the zero section of $\Lambda^2 T^*\mathbb{R}^3$.

Another, while equivalent, way to go about this: Note that

$$\Gamma(\Lambda^k(T^*\mathbb{R}^n)) \cong \Lambda^k(\Gamma(T^*\mathbb{R}^n))$$

where on the left, the exterior power is over $\mathbb{R}$, while on the right, it is over $C^\infty(\mathbb{R}^n)$. This is another way to reason out that $dy \wedge dx = -dx \wedge dy$.

(b) **Compute $\iota_X \alpha$.** ι_X is defined pointwise. That is,

$$(\iota_X \alpha)(p) := \iota_{X(p)} \alpha(p) = \alpha(p)(X(p)).$$

And $dx(\frac{\partial}{\partial x}) = \frac{\partial x}{\partial x} = 1$. Hence we have

$$\iota_X \alpha = x^2 dx(X) = x^2(x) = x^3.$$

One important point to note is that the value of $\iota_X \alpha$ at a point p does not depend at all on how X or α behave in a neighborhood of p, but only on their respective *values* at p.

One way to say this is that the map

$$\iota : \Gamma(TM) \times \Omega^k(M) \to \Omega^{k-1}(M)$$

is bilinear over $C^\infty(M)$, not just over $\mathbb{R}$. This is very useful and special. For instance, the Lie bracket is far from being bilinear over $C^\infty(M)$.

(c) **Compute $\iota_Y \beta$.** We see by linearity that

$$\iota_Y(\beta) = xyz\, \iota_Y(dx \wedge dy) + z\, \iota_Y(dy \wedge dz).$$

The easiest way to compute ι is to pass through the isomorphism between $\Lambda(V^\vee)$ and $(\Lambda(V))^\vee$. Then

$$(dx \wedge dy)(Y, -) = dx(Y)dy(-) - dy(Y)dx(-)$$

and

$$(dy \wedge dz)(Y, -) = dy(Y)dz(-) - dz(Y)dy(-).$$

Note that $dx(Y) = dz(Y) = 0$ while $dy(Y) = x^2$. Hence

$$\iota_Y(\beta) = xyz(-x^2)dx + z\,x^2dz = -x^3yzdx + x^2zdz.$$

Contracting by a vector field with only $\frac{\partial}{\partial y}$ component, note, killed off the dy components of the differential form.

(d) **Compute $\iota_Y\eta$.** This gets more annoying if you use the isomorphism from $\Lambda(V^\vee)$ to $(\Lambda(V))^\vee$ directly. Instead, recall the formula from class that

$$(\alpha \wedge \beta)(v_1, \ldots, v_k) = \sum_{\pi \in k,l\text{-shuffles}} \alpha(v_{\pi(1)}, \ldots, v_{\pi(k)})\beta(v_{\pi(k_1)}, \ldots, v_{\pi(k+l)}).$$

Since Y only has a $\frac{\partial}{\partial y}$ component, let's take $\alpha = dy$ and $\beta = dz \wedge dx$, so that $\eta = \alpha \wedge \beta$. Then, given any two vector fields V_2, V_3, we compute

$$(\alpha \wedge \beta)(Y, V_2, V_3)$$

by using the above formula. We see that the β factor is always zero if the shuffle π sends 1 to either 2 or 3. (After all, $dz \wedge dx$ evaluated on two vectors is zero if one of the vectors has no $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial z}$ component.) So the only terms in the summation that are potentially non-zero are those that come from $(1,2)$-shuffles that fix 1—i.e., with $\pi(1) = 1$. But the only such shuffle is the identity shuffle by definition of shuffle. Hence

$$(\alpha \wedge \beta)(Y, V_2, V_3) = \alpha(Y)\beta(V_2, V_3) = x^2\beta(V_2, V_3).$$

Since V_2, V_3 were arbitrary, we conclude that

$$\iota_Y\eta = x^2dz \wedge dx.$$

(e) **Compute $\alpha \wedge \beta$.** The answer is

$$(x^2z + x^2y^2z)\,dx \wedge dy \wedge dz.$$

(f) **Compute $\beta \wedge \alpha$.** The answer is

$$-\,(x^2z + x^2y^2z)\,dx \wedge dy \wedge dz.$$

(g) **Compute $\mathcal{L}_X\alpha$.** We compute

$$\begin{aligned}
\mathcal{L}_X(x^2dx + y^2dy + xydz) &= \mathcal{L}_X(x^2)dx + x^2\mathcal{L}_X(dx) + \\
&\quad \mathcal{L}_X(y^2)dy + y^2\mathcal{L}_X(dy) + \\
&\quad \mathcal{L}_X(xz)dz + xy\mathcal{L}_X(dz) && (8.1)\\
&= \mathcal{L}_X(x^2)dx + x^2d\mathcal{L}_X(x) + \\
&\quad \mathcal{L}_X(y^2)dy + y^2d\mathcal{L}_X(y) + \\
&\quad \mathcal{L}_X(xz)dz + xyd\mathcal{L}_X(z) && (8.2)\\
&= 2x^2dx + x^2d(x) + \\
&\quad 0dy + y^2d(0) + \\
&\quad xzdz + xyd(0) && (8.3)\\
&= 2x^2dx + x^2dx + xzdz \\
&= 3x^2\,dx + xz\,dz.
\end{aligned}$$

where in (8.1) we used that $\mathcal{L}_X$ is a derivation for any X. In (8.2) we use that it commutes with d. Then in (8.3) we use that for functions, $\mathcal{L}_X(f) = df(X) = X(f)$, i.e., one computes the directional derivative of f in the direction of X. Since $X = x\frac{\partial}{\partial x}$, this entails computing $\frac{\partial f}{\partial x}$, then multiplying the result by x.

(h) **Compute $\mathcal{L}_Y\beta$.** We compute

$$\begin{aligned}
\mathcal{L}_Y(xyzdx\wedge dy + zdy\wedge dz) &= \mathcal{L}_Y(xyz)dx\wedge dy + \mathcal{L}_Y(z)dy\wedge dz + \\
&\quad xyz\,\mathcal{L}_Y(dx)\wedge dy + z\,\mathcal{L}_Y(dy)\wedge dz + \\
&\quad xyz\,dx\wedge\mathcal{L}_Y(dy) + zdy\wedge\mathcal{L}_Y(dz) && (8.4)\\
&= \mathcal{L}_Y(xyz)dx\wedge dy + \mathcal{L}_Y(z)dy\wedge dz + \\
&\quad xyz\,d(\mathcal{L}_Y(x))\wedge dy + z\,d(\mathcal{L}_Y(y))\wedge dz + \\
&\quad xyz\,dx\wedge d(\mathcal{L}_Y(y)) + zdy\wedge d(\mathcal{L}_Y(z)) && (8.5)\\
&= x^3zdx\wedge dy + 0 + \\
&\quad 0 + z\cdot 2xdx\wedge dz + \\
&\quad xyz\,dx\wedge 2xdx + 0 && (8.6)\\
&= x^3zdx\wedge dy + 2xzdx\wedge dz. && (8.7)
\end{aligned}$$

Here, (8.4) again uses that $\mathcal{L}_X$ is a derivation. (8.5) uses that $\mathcal{L}_X$ commutes with d. (8.6) computes d of $\mathcal{L}_Y(f)$ using that $\mathcal{L}_Y(f) = Y(f)$. (8.7) uses that $dx\wedge dx = 0$.

(i) **Compute** $d\alpha$**.** Set $x = x^1, y = x^2, z = x^3$. Then

$$\begin{aligned} d\alpha &= \sum_i \frac{\partial(x^2)}{\partial x^i} dx^i \wedge dx + \sum_i \frac{\partial(y^2)}{\partial x^i} dx^i \wedge dy + \sum_i \frac{\partial(xy)}{\partial x^i} dx^i \wedge dz \\ &= 2x\, dx \wedge dx + 2y\, dy \wedge dy + y\, dx \wedge dz + x\, dy \wedge dz \\ &= 0 + 0 + y\, dx \wedge dz + x\, dy \wedge dz \\ &= y\, dx \wedge dz + x dy \wedge dz. \end{aligned}$$

(j) **Compute** $d\beta$**.** Set $x = x^1, y = x^2, z = x^3$. Then

$$\begin{aligned} d\beta &= \sum_i \frac{\partial(xyz)}{\partial x^i} dx^i \wedge dx \wedge dy + \sum_i \frac{\partial(z)}{\partial x^i} dx^i \wedge dy \wedge dz \\ &= yz\, dx \wedge dx \wedge dy + xz\, dy \wedge dx \wedge dy + xy\, dz \wedge dx \wedge dy + dz \wedge dy \wedge dz \\ &= 0 + 0 + xy\, dz \wedge dx \wedge dy + 0 \\ &= xy\, dx \wedge dy \wedge dz. \end{aligned}$$

(k) **Compute** $[X, Y]$**.** Recall that if

$$X = \sum X^i \frac{\partial}{\partial x^i}, \qquad Y = \sum Y^i \frac{\partial}{\partial x^i},$$

then

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

So we have

$$\begin{aligned} [x\frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial y}] &= x \frac{\partial(x^2)}{\partial x} \frac{\partial}{\partial y} - x^2 \frac{\partial(x)}{\partial y} \frac{\partial}{\partial x} \\ &= 2x^2 \frac{\partial}{\partial y}. \end{aligned}$$