Lecture 1: Motivation for linear algebra

Tuesday, August 25, 2015 9:30 AM

Admin: Textbook, syllabus, homework, midterms, final, grading, office hours, ...

MOTIVATION FOR LINEAR ALGEBRA

- Linear transformations are everywhere!
  - Calculus: integration and differentiation are linear
  - Signals: Fourier transform is linear
  - Quantum physics: time evolution is linear

- Many applications
  - Solving systems of linear equations

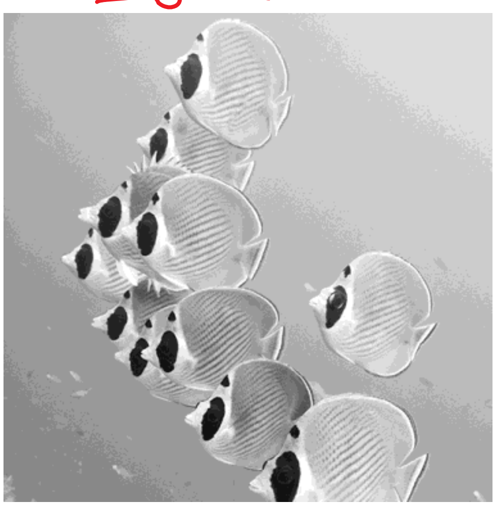
$$\begin{cases} 2x - y = 3 \\ -x + y = -2 \end{cases}$$

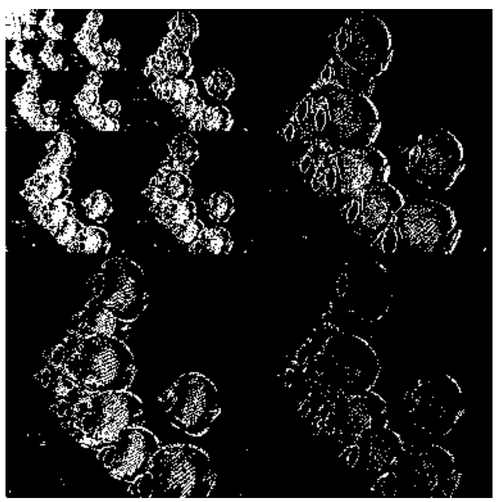
  - Solving differential equations and recursions

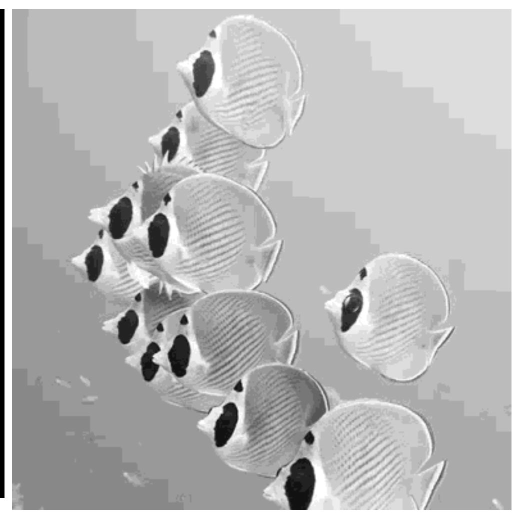
$$\ddot{y} = \dot{y} + y \qquad x_n = x_{n-1} + x_{n-2}$$

  - Image compression

Original | Keep 10% of coeffs | Result

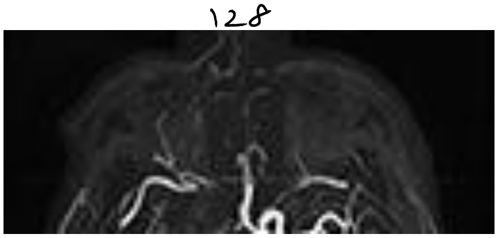






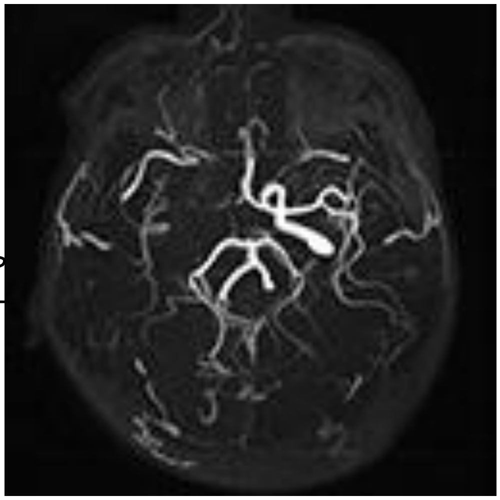
  - Compressed sensing

128



\# of variables

$N = 128^2 = 16384$

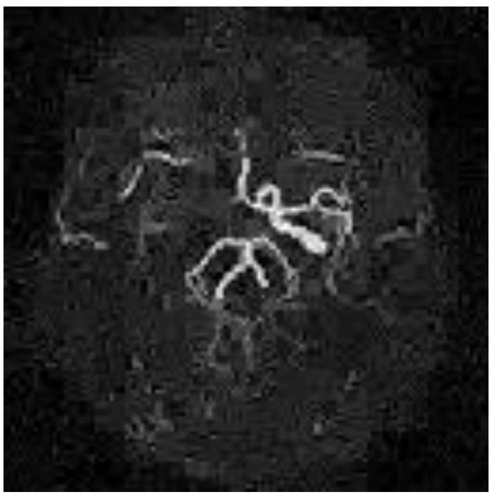


128

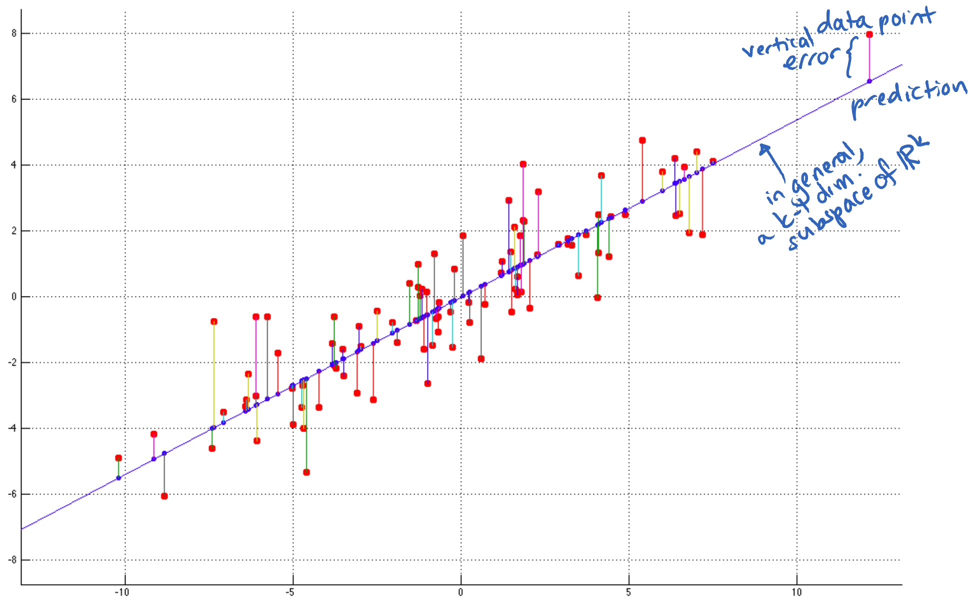
# of variables
$N = 128^2 = 16384$

# of observations
$d = 4480 = 0.27N$

15 minutes
in ℓ_1 magic

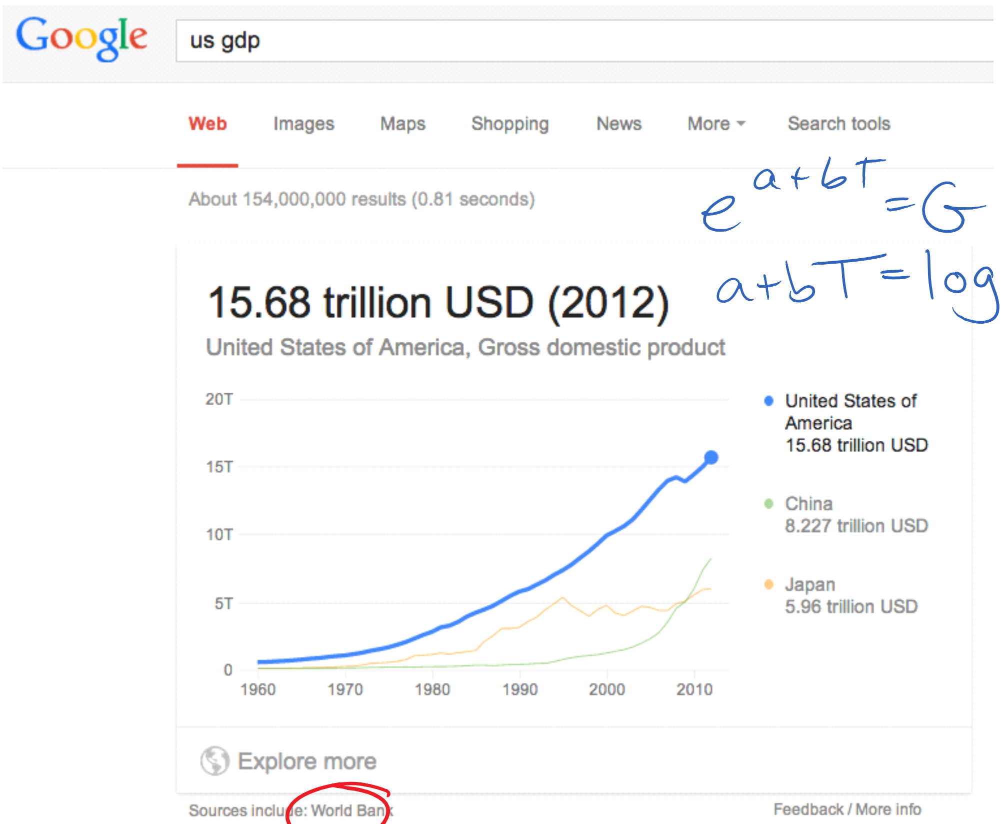


- Machine learning and statistics:
  dimension reduction, least-squares fitting



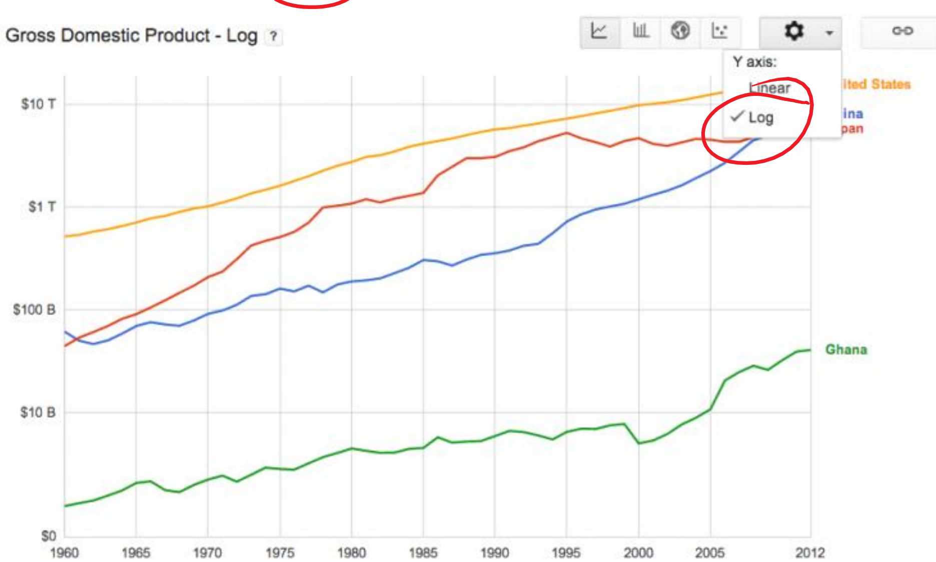
Example: Predict US gross domestic product (GDP) in 2050.

Answer:



$$e^{a+bT} = G$$

$$a + bT = \log G$$

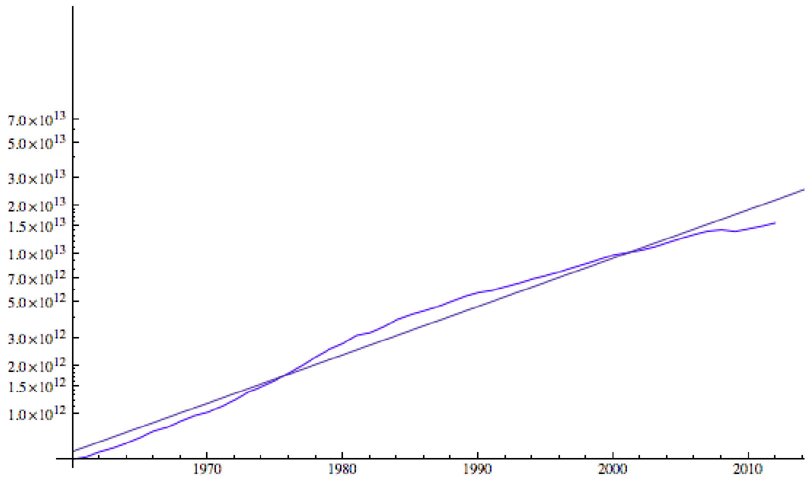


An exponential should fit the data better than a straight line.

```
USGDPdata = [1960, 5.20531e11; 1961, 5.39051e11; 1962, 5.79748e11; 1963, 6.1167e11; 1964, 6.56912e11;
1965, 7.12082e11; 1966, 7.80761e11; 1967, 8.25056e11; 1968, 9.01456e11; 1969, 9.73385e11; 1970,
1.0248e12; 1971, 1.1131e12; 1972, 1.225e12; 1973, 1.3693e12; 1974, 1.4859e12; 1975, 1.6234e12; 1976,
1.8091e12; 1977, 2.0136e12; 1978, 2.276e12; 1979, 2.5435e12; 1980, 2.7675e12; 1981, 3.1038e12; 1982,
3.2277e12; 1983, 3.5069e12; 1984, 3.9004e12; 1985, 4.1848e12; 1986, 4.425e12; 1987, 4.6989e12; 1988,
5.0619e12; 1989, 5.4397e12; 1990, 5.7508e12; 1991, 5.9307e12; 1992, 6.2618e12; 1993, 6.5829e12; 1994,
6.9933e12; 1995, 7.3384e12; 1996, 7.7511e12; 1997, 8.2565e12; 1998, 8.741e12; 1999, 9.301e12; 2000,
9.8988e12; 2001, 1.02339e13; 2002, 1.05902e13; 2003, 1.10893e13; 2004, 1.17978e13; 2005, 1.25643e13;
2006, 1.33145e13; 2007, 1.39618e13; 2008, 1.42193e13; 2009, 1.38983e13; 2010, 1.44194e13; 2011,
1.49913e13; 2012, 1.56848e13];
```

```
>> years = USGDPdata(:,1);
A = [ones(length(years),1), years];
USFit = pinv(A) * log(USGDP)

USFit =

    -1.096282320582321e+02
     6.975400397258215e-02

>> exp(USFit' * [1; 2050])

ans =

      3.099636305012074e+14
```

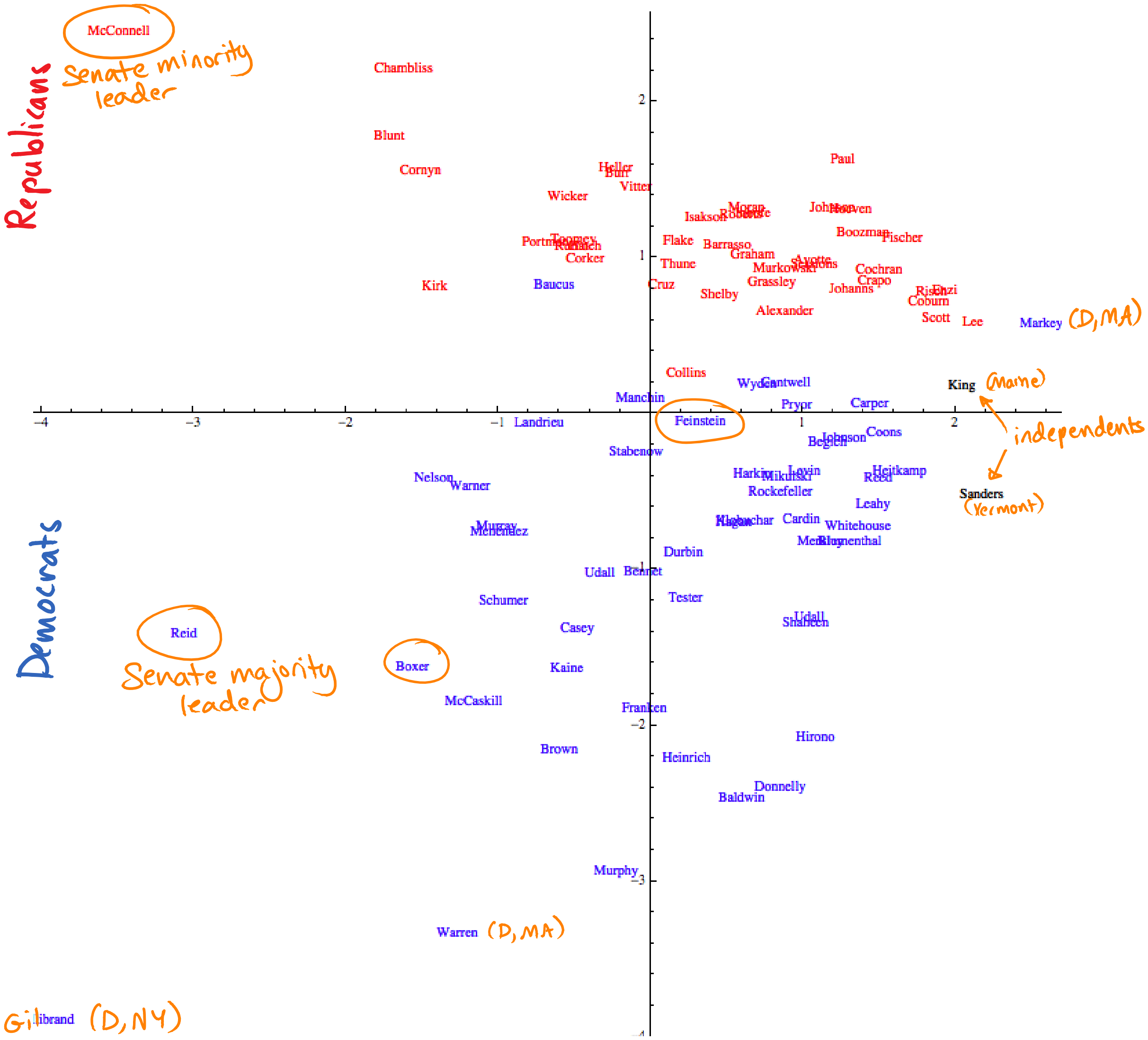
= 30 trillion dollars



```
ListPlot[{#} & /@ dataprojectedmanually, AspectRatio → 1, PlotMarkers → data[[All, 1, 1]],
 PlotStyle → (data[[All, 1, 3]]) /. {"D" → Blue, "R" → Red, "I" → Black}]
```

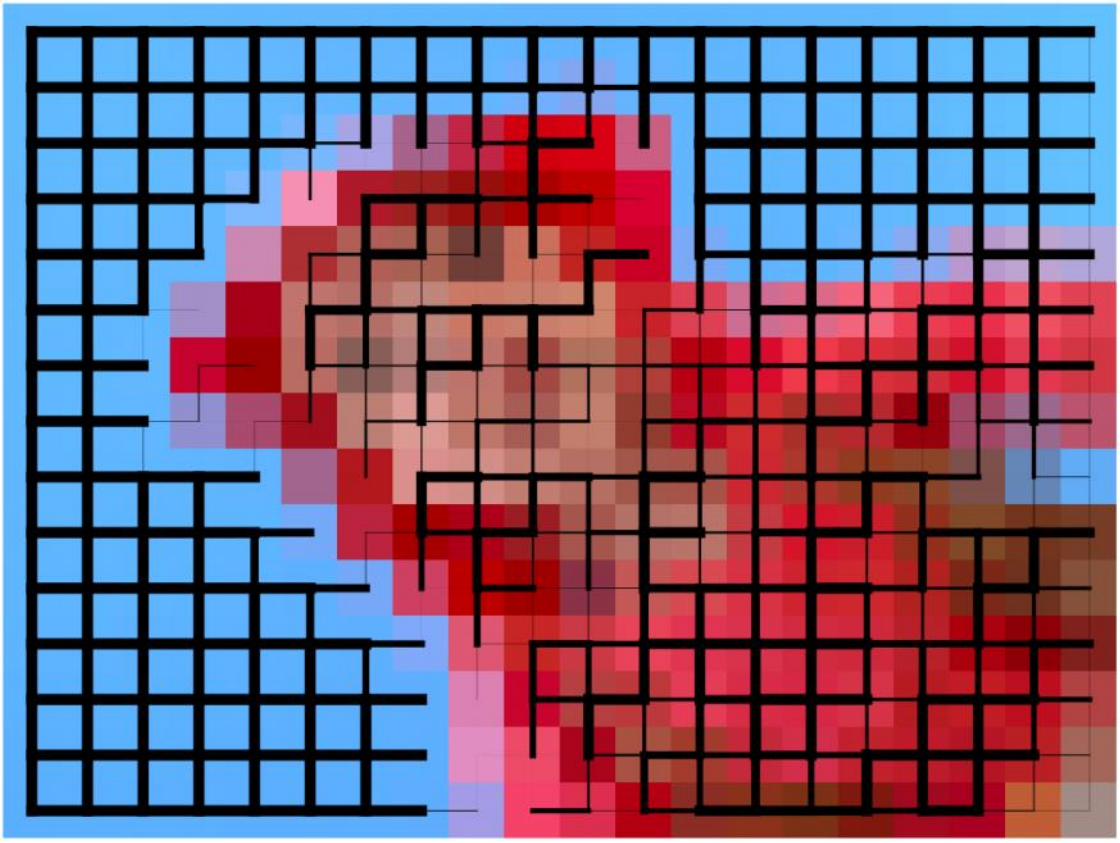
{97, 17}

{17, 2}

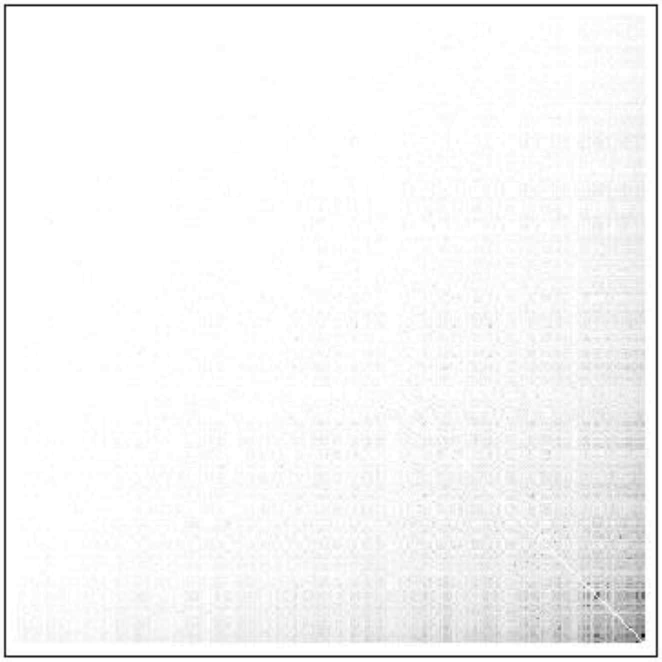


- Even purely combinatorial applications
  1. spectral clustering

APPLICATION 1: SPECTRAL IMAGE SEGMENTATION



```
matrix = Import["/Users/breic/Desktop/adjacencymatrix.txt", "Table"];
matrix += Transpose[matrix];
matrix // ArrayPlot
```



```
laplacian = DiagonalMatrix[Plus @@ # & /@ matrix] - matrix // N;
di = DiagonalMatrix[(1 / Plus @@ #) & /@ matrix // N];
evs = Eigensystem[√di .laplacian.√di] // Transpose // Reverse;

coordinates = evs[[2 ;; 9, 2]] // Transpose;

numclusters = 16;

ClusteringComponents[coordinates, numclusters, 1, Method → "PAM"]
```

embedding into $\mathbb{R}^d$

```
{Indiana Jones and the L      A Walk to Remember          Bend It Like Beckham       Con Air
 Lord of the Rings: The       Coyote Ugly                 Bridget Jones's Diary      Double Jeopardy
 Lord of the Rings: The       Dirty Dancing               Frida                      Gone in 60 Seconds
 Lord of the Rings: The       How to Lose a Guy in 10     Life Is Beautiful          Independence Day
 Raiders of the Lost Ark ,    Maid in Manhattan           Love Actually              Lethal Weapon 4
 Star Wars: Episode IV:       Pretty Woman              , Moulin Rouge             , Men in Black II          ,
 Star Wars: Episode VI:       Sister Act                  My Big Fat Greek Weddin    Pearl Harbor
 Star Wars: Episode V: T      The Princess Diaries 2:     Pride and Prejudice        The Fast and the Furiou
 The Lord of the Rings:       The Princess Diaries (W     Rabbit-Proof Fence         The Patriot
                              The Wedding Planner         Shakespeare in Love        Tomb Raider
                              What Women Want             Whale Rider                Twister

 12 Angry Men                 A Bug's Life                Amelie                     2001: A Space Odyssey
 Airplane!                    Breakfast at Tiffany's      American Beauty            All the President's Men
 American Pie                 City of Angels              Being John Malkovich       Blade Runner
 American Pie 2               Ever After: A Cinderell     Crouching Tiger            Gandhi
 Austin Powers in Goldme      Finding Nemo (Widescree     Election                   Jaws
 Austin Powers: Internat      Grease                      Eternal Sunshine of the    L.A. Confidential
 Austin Powers: The Spy       Harry Potter and the Ch     High Fidelity              Lawrence of Arabia
 Interview with the Vamp ,    Harry Potter and the Pr ,   Lock                     , Lord of the Rings: The
 Liar Liar                    Harry Potter and the So     Lost in Translation        One Flew Over the Cucko ,
 Meet the Parents             Runaway Bride               Magnolia                   Seven Samurai
 Ransom                       The Lion King: Special      Run Lola Run               The Aviator
 Spaceballs                   The NeverEnding Story       Rushmore                   The Exorcist
 Spider-Man                   The Princess Bride          Sideways                   The Godfather
 Wayne's World                The Sound of Music          The Royal Tenenbaums       The Godfather
                              Willy Wonka & the Choco     Y Tu Mama Tambien          The Graduate
                                                                                     The Great Escape
                                                                                     The Maltese Falcon
```

Cold Mountain
Collateral
Crash
Fahrenheit 9/11
Finding Neverland
Hotel Rwanda
Man on Fire
Master and Commander: T
Million Dollar Baby
Ocean's Twelve
Ray
Road to Perdition
Runaway Jury
Seabiscuit
The Manchurian Candidat
The Notebook
The Phantom of the Oper
The Pianist

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A Fish Called Wanda
Alien: Collector's Edit
Back to the Future
Back to the Future Part
Batman
Die Hard 2: Die Harder
Die Hard With a Vengean
Goldfinger
Groundhog Day
Indiana Jones and the T
Men in Black
Mission: Impossible
Predator: Collector's E
Rocky
Speed
Star Trek II: The Wrath
Terminator 2: Extreme E
The Hunt for Red Octobe
The Terminator
True Lies

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A Knight's Tale
Ice Age
Jurassic Park
Lara Croft: Tomb Raider
Minority Report
Pirates of the Caribbea
Rush Hour
Rush Hour 2
Sleeping Beauty: Specia
Spider-Man 2
Star Wars: Episode II:
Star Wars: Episode I: T
Terminator 3: Rise of t
The Fifth Element
The Incredibles
The Matrix
The Matrix: Reloaded
The Matrix: Revolutions
The Mummy
The Mummy Returns
X2: X-Men United
X-Men

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Adaptation
A Few Good Men
Air Force One
Armageddon
Clear and Present Dange
Crimson Tide
Enemy of the State
Entrapment
High Crimes
In the Line of Fire
Lethal Weapon
Lethal Weapon 2
Lethal Weapon 3
Patriot Games
Rules of Engagement
Swordfish
The Bone Collector
The Client
The Fugitive
The Negotiator
The Pelican Brief
The Rock
The Sum of All Fears

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12 Monkeys
Almost Famous
American History X
Anchorman: The Legend o
Donnie Darko
Garden State
GoodFellas: Special Edi
Grosse Pointe Blank
Heat: Special Edition
Kill Bill: Vol. 1
Kill Bill: Vol. 2
Memento
Napoleon Dynamite
Office Space
Pulp Fiction
Requiem for a Dream
Reservoir Dogs
Seven
Sin City

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Ace Ventura: Pet Detect
A League of Their Own
A River Runs Through It
Basic Instinct
Cheaper by the Dozen
Daddy Day Care
Erin Brockovich
Face/Off
Father of the Bride
Kindergarten Cop
Legally Blonde
Mrs. Doubtfire
Notting Hill
Pay It Forward
Phenomenon
Serendipity
Shall We Dance?
Sleepless in Seattle
Steel Magnolias

,

50 First Dates
Anger Management
Bad Boys II
Behind Enemy Lines
Bruce Almighty
Dodgeball: A True Under
Harold and Kumar Go to
Hero
Hidalgo
Hitch
Hostage
I Robot
Meet the Fockers
National Treasure
Ocean's Eleven
Sahara
Shrek 2
The Bourne Identity
The Bourne Supremacy
The Count of Monte Cris

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Apollo 13
As Good as It Gets
Black Hawk Down
Boys Don't Cry
Cast Away
Chocolat
Dances With Wolves: Spe
Dead Man Walking
Driving Miss Daisy
Enemy at the Gates
E.T. the Extra-Terrestr
Field of Dreams
Forrest Gump
Fried Green Tomatoes
Gladiator
Glory
Good Will Hunting
Jerry Maguire
Moonstruck
My Cousin Vinny
October Sky
Philadelphia
Primal Fear
Rain Man
Remember the Titans

2. algorithms based on fast SDD solvers

* max-flow & multi-commodity flow problems
  - for decades, best algorithms were deterministic, combinatorial
    [Goldberg-Rao '98]: $\tilde{O}(m\sqrt{n}/\varepsilon)$ for $(1-\varepsilon)$-approx max flow
    augmenting paths
    blocking flows...
  - recent breakthroughs based on numerical linear algebra, spectral graph theory
    [S-H Teng et al. '11] : $\tilde{O}(mn^{1/3}/\varepsilon^{11/3})$
    USC
    [Kelner et al., '13; Sherman '13]:
    $\tilde{O}(m/\varepsilon^2)$ or $\tilde{O}(km/\varepsilon^2)$ for k flows

* generating random spanning trees
* graph sparsification
* sparsest cut
* distributed routing

This week: Solving systems of linear equations

Linear systems of equations

* Examples, non-examples
* General problem
* Solving by elimination — complexity analysis
* Correspondence to matrices
* Solving by computer

Bigger examples

When does a system of equations have a solution?
(# variables vs. # consistent, indep. equations)

Compressed sensing

Extensions: Stability, solving repeatedly,

Geometry Systems with a special form (eg., sparse)

Examples:

$\{\ x = 3$ — 1 linear equation in 1 variable

$\{\ 2x - y = 3$ — 1 linear equation in 2 variables

$$\begin{cases} 2x - y = 3 \\ x - y = 2 \\ x + y = 0 \end{cases}$$

3 linear equations, 2 variables

$$\begin{cases} x^2 + xy = 2 \\ x + y = 3 \end{cases}$$

quadratic equation — not linear!

$$\begin{cases} x + y + z = 0 \\ x + \cos(y) + z = 2 \end{cases}$$

transcendental equation — not linear!

General problem:

Given: The system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

(m equations in n unknowns $x_1, \ldots, x_n$)

Goal: Solve for the unknowns $x_1, \ldots, x_n$

(- If there isn't a solution, then say so.
- If there are multiple solutions, find them all!)

Note: We will see later that there is always either
* one unique solution,
* no solution, or
* infinitely many solutions

Example: Solve

$$\begin{aligned} 2x - y &= 3 \\ x - y &= 2 \\ x + y &= 0 \end{aligned}$$

Answer:

Adding 3rd equation to 1st equation cancels out y, giving

$3x = 3 \quad \Rightarrow x = 1$

Substituting this back into any other equation gives $y = -1$

$\Rightarrow \boxed{(x, y) = (1, -1)}$ is the solution. ✓

(Check this.)

Example: Solve

$$\begin{aligned} x + 3y - z &= 4 \\ 3x + 10y &= 8 \end{aligned}$$

Answer: To eliminate x, subtract 3 times the first equation from the 2nd one:

$$\Rightarrow \begin{cases} x + 3y - z = 4 \\ y + 3z = -4 \end{cases}$$

$\Rightarrow y = -4 - 3z$, where z is arbitrary

Substituting into the first equation,

$$
\begin{aligned}
x &= 4 - 3y + z \\
&= 4 - 3(-4-3z) + z \\
&= 16 + 10z
\end{aligned}
$$

$\Rightarrow$ Infinitely many solutions, all of the form

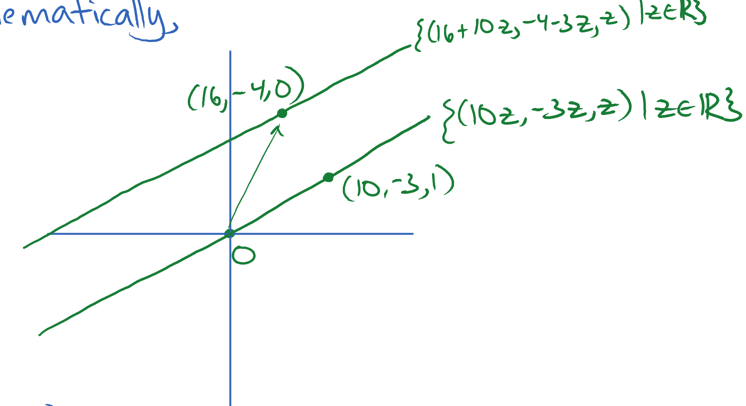
$$\boxed{x = 16 + 10z, \quad y = -4-3z, \quad z \in \mathbb{R} \text{ arbitrary}}$$

As a vector,

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 16 \\ -4 \\ 0 \end{pmatrix}}_{\text{constant coefficients}} + z\begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix}, \quad \underbrace{z}_{\text{free variable}} \in \mathbb{R}}$$

Observe:

① The set of points
$$\{(10z, -3z, z) \mid z \in \mathbb{R}^3\}$$
is a **line** through the origin $(0,0,0)$.
Adding $(16,-4,0)$ **shifts** this line.
Schematically,



② $(16,-4,0)$ is one solution to
$$
\begin{aligned}
x + 3y - z &= 4 \\
3x + 10y &= 8
\end{aligned}
$$
while $(10,-3,1)$ solves
$$
\left.\begin{aligned}
x + 3y - z &= 0 \\
3x + 10y &= 0
\end{aligned}\right\} \text{"homogeneous equations"}
$$
$\Rightarrow$ The general solution for a set of nonhomogeneous equations is a particular solution plus the general solution to the homogeneous equations.

③ We didn't have to eliminate x first...

More goals: * Solve approximately, instead of exactly

More goals: * Solve approximately, instead of exactly

* How good is the solution? (stability)
* Solve the same system multiple times (different r.h.s.s)
* Perturbed systems
* Systems with a special form
* Understand both how the algorithms work and how to use them! (and when to use them)

Example: polynomial interpolation

2 points determine a line

3 points determine a quadratic curve

⋮

any $n-1$ points $(x_1, y_1), \dots, (x_{n-1}, y_{n-1})$ determine a polynomial of degree $\leq n$

① Find the equation $y = mx + b$ for the line passing through the points

$$(1,1) \quad \text{and} \quad (6,5)$$

Answer: slope $m = \frac{5-1}{6-1} = \frac{4}{5}$

y-intercept $b = \frac{1}{5}$

(check this)

② Find the equation $y = ax^2 + bx + c$ for the parabola that goes through the points

$$(1,1), \quad (6,5), \quad (-2,3)$$

Answer: Set up three equations:

$$1 = a \cdot 1^2 + b \cdot 1 + c = a + b + c$$
$$5 = a \cdot 6^2 + b \cdot 6 + c = 36a + 6b + c$$
$$3 = a \cdot (-2)^2 + b(-2) + c = 4a - 2b + c$$

$$\left\{ \begin{array}{l} a + b + c = 1 \\ -30b - 35c = -31 \\ -6b - 3c = -1 \end{array} \right\} \longrightarrow \left\{ \begin{array}{r} a + b + c = 1 \\ 6b + 3c = 1 \\ 20c = 26 \end{array} \right\}$$

$\Rightarrow c = 13/10$

Substituting into 2nd equation,

$$b = \frac{1}{6} \cdot \left(1 - 3 \cdot \frac{13}{10}\right) = -\frac{29}{60}$$

Substituting into 1st equation,

$$a = 1 - b - c = 1 - \frac{13}{10} + \frac{29}{60} = \frac{11}{60}$$

Substituting into 1st equation,

$a = 1 - b - c = 1 - \frac{13}{10} + \frac{29}{60} = \frac{11}{60}$

$$\Rightarrow \boxed{(a,b,c) = \left(\frac{11}{60}, \frac{-29}{60}, \frac{13}{10}\right)}$$

check this. ✓

Solution method: Gaussian elimination

add multiples of the 1st equation to other equations in order to eliminate one variable from them

add multiples of 2nd equations to following equations to eliminate the next variable

⋮

with n equations in n unknowns, end up with equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a'_{33}x_3 + \cdots + a'_{3n}x_n &= b'_3 \\
&\ \ \vdots \\
a'_{n-1,n-1}x_{n-1} + a'_{n-1,n}x_n &= b'_{n-1} \\
a'_{nn}x_n &= b'_n
\end{aligned}$$

Now work backwards

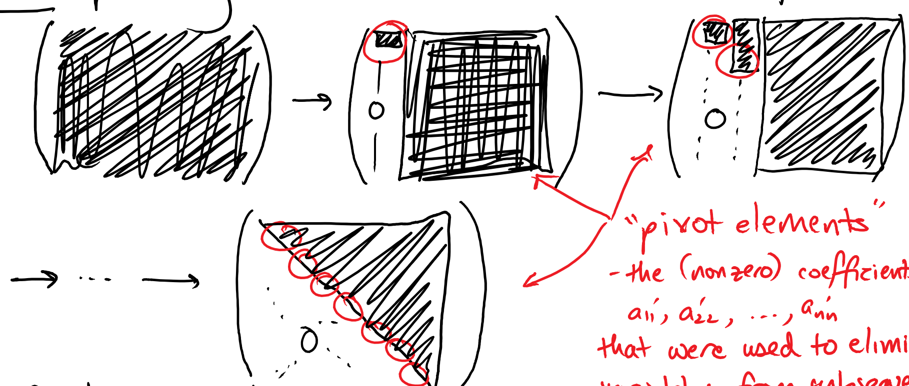
last equation $\Rightarrow x_n = \frac{b'_n}{a'_{nn}}$

substitute into eq. $n-1$ $\Rightarrow a'_{n-1,n-1}x_{n-1} = b'_{n-1} - a'_{n-1,n}\frac{b'_n}{a'_{nn}}$

Keep on substituting backwards, solving for variables

$x_n, x_{n-1}, x_{n-2}, \ldots, x_2, x_1$

Picture: possibly nonzero entries in the matrix of coefficients



"pivot elements"
- the (nonzero) coefficients $a'_{11}, a'_{22}, \ldots, a'_{nn}$ that were used to eliminate variables from subsequent equations

Back substitution...

Complexity analysis

Claim: G.E. on an $n \times n$ system of equations uses $O(n^3)$ basic operations (multiplication, division, addition, etc.)

division, addition, etc

Proof sketch:

Eliminating x_1 using the first equation, you need to update $n-1$ other equations, each with n coefficients $\Rightarrow O(n^2)$ steps

Repeating for the other variables $\Rightarrow O(n^3)$ conservatively

Back-substitution takes $O(n^2)$ steps. Why? $\boxtimes$

Example

Solve
$$\begin{cases} -x_1 + 3x_2 - 2x_3 = 1 \\ -x_1 + 4x_2 - 3x_3 = 0 \\ -x_1 + 5x_2 - 4x_3 = 0 \end{cases}$$

$$\longrightarrow \begin{pmatrix} & 3 & -2 \\ & 1 & -1 \\ & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} & 3 & -2 \\ & 1 & -1 \\ & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Stuck! 3^{rd} equation is $0x_1 + 0x_2 + 0x_3 = 1$
$\Rightarrow$ no solution.

Solving linear equations on your computer

① Mathematica:

A = {{1, 1, 1}, {36, 6, 1}, {4, -2, 1}};
b = {1, 5, 3}
use LinearSolve[A, b]

② Matlab/Octave:

```
>> A = [1 1 1; 36 6 1; 4 -2 1]

A =

     1     1     1
    36     6     1
     4    -2     1

>> b = [1; 5; 3]

b =

     1
     5
     3

>> x = A \ b

x =
```

```
>> A = [1 1 1; 36 6 1; 4 -2 1]

A =

     1     1     1
    36     6     1
     4    -2     1

>> b = [1; 5; 3]

b =

     1
     5
     3

>> x = A \ b

x =

    0.1833
   -0.4833
    1.3000

>> A*x - b

ans =

   1.0e-15 *

         0
    0.8882
    0.4441
```

(see also "format rat")

(what if there's no solution?)

③ Numpy in python:

```
>>> import numpy as np
>>> A = np.array([[1,1,1],[36,6,1],[4,-2,1]])
>>> b = np.array([1,5,3])
>>> x = np.linalg.solve(A,b)
>>> x
array([ 0.18333333, -0.48333333,  1.3       ])
>>> np.dot(A,x) - b
array([  0.00000000e+00,   0.00000000e+00,   4.44089210e-16])

>>> n = 800; A = np.random.randn(n,n); b = np.dot(A, np.random.randn(n));
>>> start = time.time(); x = np.linalg.solve(A,b); end = time.time(); end-start
0.029742002487182617
>>> n = 1600; A = np.random.randn(n,n); b = np.dot(A, np.random.randn(n));
>>> start = time.time(); x = np.linalg.solve(A,b); end = time.time(); end-start
0.17376208305358887
>>> n = 3200; A = np.random.randn(n,n); b = np.dot(A, np.random.randn(n));
>>> start = time.time(); x = np.linalg.solve(A,b); end = time.time(); end-start
1.0949180126190186
>>> n = 6400; A = np.random.randn(n,n); b = np.dot(A, np.random.randn(n));
>>> start = time.time(); x = np.linalg.solve(A,b); end = time.time(); end-start
7.7281270027160645
```