Homework 1

Thursday, August 27, 2015 9:30 AM

Due in class September 3^{rd}

Note: For full credit, show your work!
You are welcome to discuss the problems with others, but write up your own solutions.

① What is the complexity of using Gaussian elimination to solve $Ax=b$, where A is an $n \times n$ **tridiagonal** matrix? Is it $O(n)$? $O(n^2)$? $O(n^3)$? Justify your answer!

② Find the degree-three polynomial $p(x)=ax^3+bx^2+cx+d$ with

$$p(1)=4 \qquad p'(-1)=0$$
$$p(2)=2 \qquad p'(2)=6$$

- First solve for p by using Gaussian elimination on a set of linear equations.
  Show your work. Please use compact matrix notation to simplify grading.
- Now enter the matrix into Matlab, Mathematica, or equivalent software, and solve the equations there. Again, show your work (commands and results).

③ By hand (not using a computer), find all solutions to the equations:

$$x_1 + x_2 + 2x_3 = 1$$
$$3x_1 + 3x_3 + 3x_4 = 6$$
$$2x_1 + 2x_2 - x_3 + x_4 = 3$$

④ By hand, compute the LU decomposition of

$$\begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}$$

(4) By hand, compute the LU decomposition of

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 0 & 6 & 7 & 8 \\ 0 & 0 & -22 & 9 \end{pmatrix}$$

(In general, it is easy to see that the L, U factors of a tridiagonal matrix are themselves tridiagonal. [See problem 3.10.6.] This makes solving $Ax=b$ using the LU decomposition very fast. On the other hand, A^{-1} can be dense — as it is in this example — so naively multiplying out $A^{-1}b$ takes quadratic time, much slower than using the LU decomposition!)

(5) By discretizing the interval and setting up a system of linear equations, solve numerically the differential equation

$$f''(t) - 2f'(t) = \cos(t)$$

on the interval $[0, \pi]$, with boundary conditions

$$f(0) = 0 \quad , \quad \underline{f(\pi) = -1}$$

After setting up the equations, solve them with sparse matrix routines in Matlab or Mathematica.
Experiment with finer discretizations.
Show your work, and plot the results

EXTRA CREDIT:

(6) Consider the differential equation

$$\frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y) = x^2 + y^2$$

on the rectangle $x \in [0,1]$, $y \in [0,1]$, with boundary conditions

$$f(0,y) = f(1,y) = f(x,0) = f(x,1) = 0$$

As in problem 5, discretize the rectangle — but this time into an $n \times n$ grid. Set up the corresponding system of linear equations. (You should have n^2 equations in n^2 variables. Try to present them in concise mathematical notation.

There is no need to solve these equations!

(Unless you want to.)

Instead of solving the equations, just sketch by hand the positions of the nonzero entries of the matrix. For example, if the matrix were tridiagonal (it's not!), you could sketch it like: