Lecture 4: Vector spaces

Thursday, September 3, 2015 9:30 AM

Admin:

VECTOR SPACES

Why? They're everywhere

4D vectors $\mathbb{R}^4$ (a, b, c, d)

cubic polynomials $ax^3 + bx^2 + cx + d$

2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(a,b,c,d) + (e,f,g,h) = (a+e, b+f, c+g, d+h)$$

$$(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + i) = (a+e)x^3 + (b+f)x^2 + \cdots$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

⇒ We should abstract their properties, study them together

(and matrices are best understood as linear transformations on vector spaces)

Definition: A vector space consists of

- a set of "vectors" V
- a field $\mathbb{F}$ (often the reals $\mathbb{R}$ or complex #s $\mathbb{C}$)
- operations of
  - vector addition $V \times V \to V$, denoted $\vec{x} + \vec{y}$
  - scalar multiplication $\mathbb{F} \times V \to V$, denoted $\alpha \vec{x}$

that satisfy:

- closure under addition & scalar multiplication:

that satisfy:

- closure under addition & scalar multiplication:

for all $\alpha \in \mathbb{F}$, $\vec{x}, \vec{y} \in V$

$$\boxed{\begin{array}{c}\vec{x}+\vec{y} \in V \\ \alpha \vec{x} \in V\end{array}}$$

- existence of $\vec{0} \in V$

$\vec{0} + \vec{x} = \vec{x}$ for all $\vec{x}$

- additive inverses

for all $\vec{x} \in V$, there exists $\vec{y} \in V$ s.t. $\vec{x}+\vec{y} = \vec{0}$

for all $\alpha, \beta \in \mathbb{F}$, $\vec{x}, \vec{y}, \vec{z} \in V$:

- commutativity $\quad \vec{x}+\vec{y} = \vec{y}+\vec{x}$
- associativity $\quad \vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$

  $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$
- distributivity $\quad (\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$

  $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$
- $1\vec{x} = \vec{x}$ (identity for multiplication)

Note: The most important properties to check are closure under addition and scalar multiplication. The other properties are usually automatic.

Examples: Vector spaces are everywhere!

① $\mathbb{R}^n$: real vectors $(x_1, x_2, \dots, x_n)$

coordinate-wise addition & multiplication

$\mathbb{C}^n$

② matrices $\mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$

③ the single-point sets $\{0\}$ or $\{(0,0,\dots,0)\}$

(trivially closed under addition & multiplication)

But these are NOT vector spaces:

$\{1\}$, $\{(1,0,0)\}$

$\{(0,0),\ (1,0)\}$
the interval $[0,1]$

(4) function spaces, eg.,
all functions $\mathbb{R} \to \mathbb{R}$
all functions $[0,1] \to \mathbb{R}$

addition $(f+g)(x) = f(x) + g(x)$
multiplication $(\alpha f)(x) = \alpha \cdot f(x)$

Subspaces!

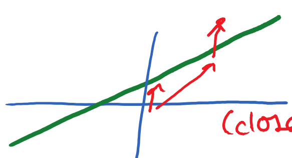
(5) $\{(\alpha, 2\alpha) : \alpha \in \mathbb{R}\}$
includes $(0,0)$, closed under $+$, $\times$ ✓

ALL subspaces of $\mathbb{R}^2$:

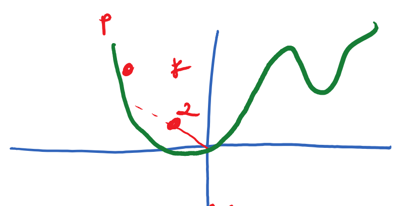
$\{0\}$, lines through 0 $\{(x,y) : ax+by=0\}$, $\mathbb{R}^2$ itself

NOT subspaces:

other lines:



curves:



(closed neither under addition or multiplication!)

Important: Lines/planes/hyperplanes that **don't** go through the origin (0) are NOT subspaces!!

(6) The SPAN of any (finite or infinite) set of points S

$$\boxed{\text{Span}(S) \text{ is defined to be the set of all finite linear combinations of elements from } S}$$

$\left(\text{i.e., all sums } \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \text{ for } \alpha_j \in \mathbb{F},\ v_j \in S \right)$

By definition, this is closed under $+$, $\times$, and hence a vector space.

Examples:

$\text{Span}\{(1,2)\}$ $\qquad$ $\text{Span}\{(1,2), (-1,-2)\}$

$\text{Span}\{(1,0), (0,1)\}$

Claim: Span(S) is the smallest vector space that contains all the points in S.

Proof:

- Let T be a vector space containing all of S.
- Let $\vec{v} \in \text{Span}(S)$.

$\Rightarrow \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r$, with all $\vec{v}_i \in S$

$\Rightarrow$ all $\vec{v}_i \in T$

$\Rightarrow$ all $\alpha_i \vec{v}_i \in T$ (closure under mult.)

$\Rightarrow \vec{v} = \sum_i \alpha_i \vec{v}_i \in T$ (closure under addition)

$\Rightarrow \text{Span}(S) \subseteq T$. ✓ $\square$

7. Polynomials $= \text{Span}\{1, x, x^2, x^3, \ldots\}$

Continuous functions

Differentiable functions

Functions f with $f(1) = 0$, $f'(2) = 0$

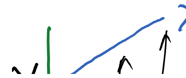
8. The SUM of two subspaces is a subspace.

<u>Definition</u>: For two subsets X and Y in a vector space V, let

$$X + Y \doteq \{x + y \mid x \in X, y \in Y\}$$

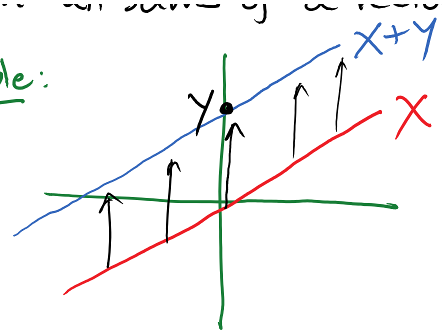
(In English: all sums of a vector in X and a vector in Y.)

Example:



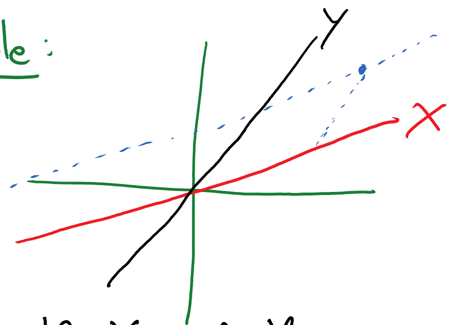
X+Y $\leftarrow$ not a subspace!

Example:



not a subspace!
(sometimes called an "affine subspace")

Example:



What is X+Y?
Answer: The whole plane!
For any other point, draw this parallelogram ✓

Claim 1: If X and Y are <u>subspaces</u>, then X+Y is a subspace.

Proof: The key properties to check are closure under addition and multiplication.

Closure under addition:

want to show (WTS) if $a, b \in X+Y$, then $a+b \in X+Y$:

$a \in X+Y \Rightarrow a = x+y$ for some $x \in X, y \in Y$

$b \in X+Y \Rightarrow b = x'+y'$ " x' " y' "

$$a+b = (x+y)+(x'+y')$$
$$= \underbrace{(x+x')}_{X} + \underbrace{(y+y')}_{Y} \in X+Y \checkmark$$

Closure under multiplication:

WTS: If $a \in X+Y$, then for all scalars α, $\alpha a \in X+Y$:

$$a = x+y$$
$$\alpha a = \alpha(x+y) = \underbrace{(\alpha x)}_{X} + \underbrace{(\alpha y)}_{Y} \checkmark$$

$\square$

Claim 2: For subsets S and T of a vector space V,
$\mathrm{Span}(S) + \mathrm{Span}(T) = \mathrm{Span}(S \cup T)$

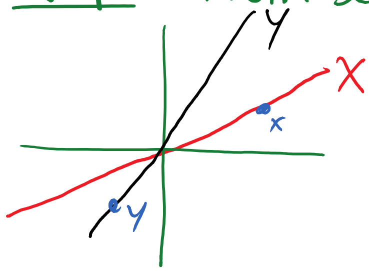
Proof: $\mathrm{Span}(S) = \{$finite linear combinations of elts of $S\}$
$\mathrm{Span}(T) = \{$ " " " " $T\}$

$\therefore \quad x \in \mathrm{Span}(S) + \mathrm{Span}(T)$

$\iff x = \sum_{j=1}^{k} \alpha_j \underset{\overset{\uparrow}{S}}{s_j} + \sum_{j=1}^{\ell} \beta_j \underset{\uparrow}{t_j}$

$\iff x$ is a finite linear combination of elements of $S \cup T$. ✓ $\square$

Example: From before



$X = \mathrm{Span}(\{x\})$

$Y = \mathrm{Span}(\{y\})$

$\mathrm{Span}(\{x, y\}) = \mathbb{R}^2 = X + Y$

Proof. To prove (4.1.1), demonstrate that the two closure properties **(A1)** and **(M1)** hold for $\mathcal{S} = \mathcal{X} + \mathcal{Y}$. To show **(A1)** is valid, observe that if $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, then $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{Y}$, and therefore $\mathbf{u} + \mathbf{v} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in \mathcal{S}$. To verify **(M1)**, observe that $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication so that $\alpha\mathbf{x}_1 \in \mathcal{X}$ and $\alpha\mathbf{y}_1 \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} = \alpha\mathbf{x}_1 + \alpha\mathbf{y}_1 \in \mathcal{S}$ for all α. To prove (4.1.2), suppose $\mathcal{S}_X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\mathcal{S}_Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$, and write

$$\mathbf{z} \in span\,(\mathcal{S}_X \cup \mathcal{S}_Y) \Longleftrightarrow \mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i + \sum_{i=1}^{t} \beta_i \mathbf{y}_i = \mathbf{x} + \mathbf{y} \text{ with } \mathbf{x} \in \mathcal{X},\ \mathbf{y} \in \mathcal{Y}$$
$$\Longleftrightarrow \mathbf{z} \in \mathcal{X} + \mathcal{Y}. \quad \blacksquare$$

Example 4.1.8

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law—sketch a picture for yourself.

Exercises for section 4.1

4.1.1. Determine which of the following subsets of $\Re^n$ are in fact subspaces of $\Re^n$ $(n > 2)$.

(a) $\{\mathbf{x} \mid x_i \geq 0\}$, (b) $\{\mathbf{x} \mid x_1 = 0\}$, (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$,

(d) $\left\{\mathbf{x} \;\middle|\; \sum_{j=1}^{n} x_j = 0\right\}$, (e) $\left\{\mathbf{x} \;\middle|\; \sum_{j=1}^{n} x_j = 1\right\}$,

(f) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},$ where $\mathbf{A}_{m\times n} \neq \mathbf{0}$ and $\mathbf{b}_{m\times 1} \neq \mathbf{0}\}$.

4.1.2. Determine which of the following subsets of $\Re^{n\times n}$ are in fact subspaces of $\Re^{n\times n}$.

(a) The symmetric matrices. (b) The diagonal matrices.
(c) ~~The nonsingular matrices.~~ (d) ~~The singular matrices.~~
? (e) The triangular matrices. (f) The upper-triangular matrices.
(g) All matrices that commute with a given matrix $\mathbf{A}$.
(h) All matrices such that $\mathbf{A}^2 = \mathbf{A}$.
(i) All matrices such that $trace\,(\mathbf{A}) = 0$.

4.1.3. If $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

$\mathbb{R}^3$

4.1.4. Why must a real or complex nonzero vector space contain an infinite number of vectors?

4.1.5. Sketch a picture in $\Re^3$ of the subspace spanned by each of the following.

(a) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -9 \\ -6 \end{pmatrix} \right\}$, line

(b) $\left\{ \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, xy-plane

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. $\Re^3$

4.1.6. Which of the following are spanning sets for $\Re^3$?

(a) $\{(1\ \ 1\ \ 1)\}$ (b) $\{(1\ \ 0\ \ 0),\ (0\ \ 0\ \ 1)\}$,

(c) $\{(1\ \ 0\ \ 0),\ (0\ \ 1\ \ 0),\ (0\ \ 0\ \ 1),\ (1\ \ 1\ \ 1)\}$,

(d) $\{(1\ \ 2\ \ 1),\ (2\ \ 0\ \ -1),\ (4\ \ 4\ \ 1)\}$, 2 first + second

(e) $\{(1\ \ 2\ \ 1),\ (2\ \ 0\ \ -1),\ (4\ \ 4\ \ 0)\}$.

4.1.7. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why $span\,(\mathcal{M} \cup \mathcal{N}) = span\,(\mathcal{M}) + span\,(\mathcal{N})$.

skip

4.1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.

(a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$.

(b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

4.1.9. For $\mathbf{A} \in \Re^{m\times n}$ and $\mathcal{S} \subseteq \Re^{n\times 1}$, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. We refer to $\mathbf{A}(\mathcal{S})$ as the set of ***images*** of $\mathcal{S}$ under $\mathbf{A}$.

(a) If $\mathcal{S}$ is a subspace of $\Re^n$, prove $\mathbf{A}(\mathcal{S})$ is a subspace of $\Re^m$.

(b) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ spans $\mathcal{S}$, show $\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k$ spans $\mathbf{A}(\mathcal{S})$.

4.1.10. With the usual addition and multiplication, determine whether or not the following sets are vector spaces over the real numbers.

(a) $\Re$, (b) $\mathcal{C}$, (c) The rational numbers.

4.1.11. Let $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\}$ and $\mathcal{N} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space. Prove that $span\,(\mathcal{M}) = span\,(\mathcal{N})$ if and only if $\mathbf{v} \in span\,(\mathcal{M})$.

4.1.12. For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, prove that $span\,(\mathcal{S})$ is the intersection of all subspaces that contain $\mathcal{S}$. **Hint:** For $\mathcal{M} = \bigcap_{\mathcal{S}\subseteq\mathcal{V}} \mathcal{V}$, prove that $span\,(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span\,(\mathcal{S})$.

(8) Examples over other fields

$\mathbb{F}_p$ = field of numbers mod p (for a prime p)
$\mathbb{F}_2 = \{0, 1\}$

Bit strings of length n form a vector space:

$n=3$: $(0,0,0), (0,0,1), (0,1,0), (0,1,1)$
$(1,0,0), (1,0,1), (1,1,0), (1,1,1)$

addition is coordinate-wise, mod 2
$(0,0,1) + (0,1,1) = (0,1,0)$

subspace, eg.,
$\mathrm{Span}(\{(0,0,1), (1,0,1)\})$
$= \{(0,0,0), (0,0,1), (1,0,1), (1,0,0)\}$

Problem:

1. How many subspaces are there of $\mathbb{R}^2$?
   Answer: Infinitely many!
   (lines through the origin)
2. How many subspaces are there of $\mathbb{R}$?
   Answer: Two! $\{0\}$ and $\mathbb{R}$ itself.
3. How many subspaces are there of $\{0,1\}^2$?

   $\{(0,0)\}$, everything $\{0,1\}^2$
   $\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$
   $\{(0,0),(1,0)\}$, $\{(0,0),(1,1)\}$
   and that's it!
   (if a subspace contains two of the nonzero points, then it also includes their sum, which is the last nonzero point $\therefore$ $(1,0)+(0,1)+(1,1) = (0,0)$ means that any two sum to the third)
   $1 + 4 + 1 = 6$
4. How many subspaces are there of $\{0,1\}^n$?
   We'll answer this later! Definitely $< \infty$ though :)

Binary operations on subspaces

$+$ sum, $\cap$ intersection, $\cancel{\cup}$ union does not generally give another vector space

Binary operations on vector spaces

$\oplus$ direct sum

$\times$ direct product

these are the same on a finite number of operands

Basically, $\mathbb{R}^m \oplus \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$

$\otimes$ tensor product

Basically,

$$V \otimes W = \text{Span}\left(\{v \otimes w \mid v \in V, w \in W\}\right)$$

(not quite a precise definition)

e.g., $(1,0)\otimes(1,0) + (0,1)\otimes(0,1) \in \mathbb{R}^2 \otimes \mathbb{R}^2$

and cannot be simplified further (to $v \otimes w$)

$$\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}$$

vector $(1,1,-1)$ and automatically contains any multiple $(c,c,-c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c & c & -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$: all attainable right-hand sides b and all solutions to $Ax = 0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

(b) The plane of vectors b with $b_1 = 1$.

(c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).

(d) All combinations of two given vectors $(1,1,0)$ and $(2,0,1)$.

(e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

3. Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

5. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all x.
4. For each x there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3,1)+(5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers, with $x+y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?

(c) Suppose $(x_1,x_2)+(y_1,y_2)$ is defined to be (x_1+y_2,x_2+y_1). With the usual $cx=(cx_1,cx_2)$, which of the eight conditions are not satisfied?

6. Let $\mathbf{P}$ be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

7. Which of the following are subspaces of $\mathbf{R}^\infty$?

(a) All sequences like $(1,0,1,0,\ldots)$ that include infinitely many zeros. NO

(b) All sequences $(x1,x2,\ldots)$ with $x_j=0$ from some point onward. YES

(c) All decreasing sequences: $x_{j+1}\le x_j$ for each j.

(d) All convergent sequences: the x_j have a limit as $j\to\infty$. ?

(e) All arithmetic progressions: $x_{j+1}-x_j$ is the same for all j.

(f) All geometric progressions $(x_1,kx_1,k^2x_1,\ldots)$ allowing all k and x_1.

8. Which of the following descriptions are correct? The solutions x of

$$Ax=\begin{bmatrix}1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

form

(a) a plane.

(b) a line.

(c) a point.

(d) a subspace.

(e) the nullspace of A.

(e) the nullspace of A.

(f) the column space of A.

9. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

10. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

11. (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

12. The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ____. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

13. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

14. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

15. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0,0,0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

16. $\mathbf{P}_0$ is the plane through $(0,0,0)$ parallel to the plane $\mathbf{P}$ in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

17. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, R^3 itself, or $\mathbf{Z}$ containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe the five types of subspaces of $\mathbf{R}^4$.

18. (a) The intersection of two planes through $(0,0,0)$ is probably a ____ but it could be a ____. It can't be the zero vector $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____ but it could be a ____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on* $x+y$ *and* cx.

19. Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.

20. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

(c) The matrices that have $(1,1,1)$ in their nullspace form a subspace.

Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of ____ is also a combination of the columns of A. Which two matrices have the same column ____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

26. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of A and AB are not equal.

27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A-I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax=b$ is solvable for every b, then $\boldsymbol{C}(A)=$ ____.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 Solving $Ax=0$ and $Ax=b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax=b$ and it was $x=-A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x=0$ (multiply $Ax=0$ by A^{-1}). The column space is the whole space ($Ax=b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x=x_p+x_n$:

$$\textbf{Complete solution} \qquad Ax_p=b \quad \textbf{and} \quad Ax_n=0 \qquad \textbf{produce} \quad A(x_p+x_n)=b.$$

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax=b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax=0$. We will find the conditions for b to lie in the column space (so that $Ax=b$ is solvable). The 1 by 1 system $0x=b$, one equation and one unknown, shows two possibilities:

$0x=b$ has *no solution* unless $b=0$. The column space of the 1 by 1 zero matrix contains only $b=0$.

$0x=0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p=0$, and the complete solution is $x=x_p+x_n=0+(\text{any } x)$.

More important examples:

SUBSPACES OF A MATRIX

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SUBSPACES OF A MATRIX

Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real-valued matrix.

- $\text{Range}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

  $= \text{Span}(\text{columns of } A)$

  AKA "column space" of A

  Observe: $\vec{b} \in \text{Range}(A) \iff A\vec{x} = \vec{b}$ has a solution

- $\text{Range}(A^T) = \text{Span}(\text{rows of } A)$

  "row space"

- $\text{Kernel}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ (solutions to the homogeneous equations)

  AKA "null space" of A