# Lecture 5: Vector spaces (continued)

Thursday, September 3, 2015 9:30 AM

**Admin:**

**Aside:**

## Matrix multiplication in practice: Strassen's algorithm

http://en.wikipedia.org/wiki/Strassen_algorithm

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Compute
$$\begin{aligned} M_1 &= (A_{11}+A_{22})(B_{11}+B_{22}), & M_5 &= (A_{11}+A_{12})B_{22} \\ M_2 &= (A_{21}+A_{22})B_{11} & M_6 &= (A_{21}-A_{11})(B_{11}+B_{12}) \\ M_3 &= A_{11}(B_{12}-B_{22}) & M_7 &= (A_{12}-A_{22})(B_{21}+B_{22}) \\ M_4 &= A_{22}(B_{21}-B_{11}) \end{aligned}$$

using 7 matrix multiplications (instead of the 8 obvious ones $A_{11}B_{11}+A_{12}B_{21}, \ldots, A_{21}B_{12}+A_{22}B_{22}$).

$$\Rightarrow AB = \begin{pmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1-M_2+M_3+M_6 \end{pmatrix}$$

Asymptotic complexity $O(n^{\log_2 7 = 2.81\ldots})$ for $n\times n$ matrices.

## Method 2: Matrix inversion

• Definition • Existence • Properties

**Definition:** If $A$ is an $n\times n$ square matrix, then $A^{-1}$ is the matrix satisfying
$$A \cdot A^{-1} = I,$$
if such a matrix exists.

**Examples:**
$$I^{-1} = I$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{-1} = \text{does not exist!}$$

**Exercise**: Prove that for a general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $A^{-1}$ exists $\iff ad - bc \neq 0$.

If $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Properties of the matrix inverse:**

- $AA^{-1} = I = A^{-1}A$

  Equivalently, $(A^{-1})^{-1} = A$

- If it exists, then $A^{-1}$ is unique.

  Proof:

  Say $X$ and $Y$ are both inverses of $A$.

  $$\begin{aligned} X = XI = X(AY) \\ &= (XA)Y \\ &= IY \\ &= Y \quad \checkmark \end{aligned}$$

- If $A$ and $B$ are invertible, so is $AB$:

  $$(AB)^{-1} = B^{-1}A^{-1}$$

  Since $(B^{-1}A^{-1})AB = B^{-1}IB = I$ ✓

  If $A, B, C$ are invertible, so is $ABC$:

  $$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

- Not all matrices are invertible!

## How to invert a matrix

**Goal:** Solve for the matrix $X$ in

$$AX = I$$

$$\Longleftrightarrow \quad \underset{3\times 3}{A} \underbrace{\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I}$$

$$\Longleftrightarrow \quad A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Use Gaussian elimination!

**Example:** $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-1/2} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) 2/3$$

r.h.s. for 1st system, 2nd system, 3rd system

means $AX = I$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 1/3 & 1/3 & 2/3 & 1 \end{array}\right) 3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/2 & 3 & 3 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right) 2/3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right) \text{divide by 2}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \underbrace{1}_{x} & \underbrace{2}_{y} & \underbrace{3}_{z} \end{array}\right)$$

means $IX = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$$\Rightarrow X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \checkmark$$

**But:** Matrix inversion is not the answer we want.

In practice, we almost **never** invert matrices!

1. Too slow.
2. Might not even exist.
3. Numerically unstable

Example:

```
>> A = [.01 0; 0 1];
inv(A)

ans =

   100     0
     0     1
```

```
>> A = [.02 0; 0 1];
inv(A)

ans =

    50     0
     0     1
```

4. Can take **sparse** matrices to **dense** matrices ($\Rightarrow$ uses too much memory!)

Example:

```
>> n = 10;
e = ones(n-1,1);
A = -2*eye(n) + diag(ones(n-1,1), -1) + diag(ones(n-1,1), 1)

A =

    -2     1     0     0     0     0     0     0     0     0
     1    -2     1     0     0     0     0     0     0     0
     0     1    -2     1     0     0     0     0     0     0
     0     0     1    -2     1     0     0     0     0     0
     0     0     0     1    -2     1     0     0     0     0
     0     0     0     0     1    -2     1     0     0     0
     0     0     0     0     0     1    -2     1     0     0
     0     0     0     0     0     0     1    -2     1     0
     0     0     0     0     0     0     0     1    -2     1
     0     0     0     0     0     0     0     0     1    -2

>> inv(A)

ans =

   -0.9091   -0.8182   -0.7273   -0.6364   -0.5455   -0.4545   -0.3636   -0.2727   -0.1818   -0.0909
   -0.8182   -1.6364   -1.4545   -1.2727   -1.0909   -0.9091   -0.7273   -0.5455   -0.3636   -0.1818
   -0.7273   -1.4545   -2.1818   -1.9091   -1.6364   -1.3636   -1.0909   -0.8182   -0.5455   -0.2727
   -0.6364   -1.2727   -1.9091   -2.5455   -2.1818   -1.8182   -1.4545   -1.0909   -0.7273   -0.3636
   -0.5455   -1.0909   -1.6364   -2.1818   -2.7273   -2.2727   -1.8182   -1.3636   -0.9091   -0.4545
   -0.4545   -0.9091   -1.3636   -1.8182   -2.2727   -2.7273   -2.1818   -1.6364   -1.0909   -0.5455
   -0.3636   -0.7273   -1.0909   -1.4545   -1.8182   -2.1818   -2.5455   -1.9091   -1.2727   -0.6364
   -0.2727   -0.5455   -0.8182   -1.0909   -1.3636   -1.6364   -1.9091   -2.1818   -1.4545   -0.7273
   -0.1818   -0.3636   -0.5455   -0.7273   -0.9091   -1.0909   -1.2727   -1.4545   -1.6364   -0.8182
   -0.0909   -0.1818   -0.2727   -0.3636   -0.4545   -0.5455   -0.6364   -0.7273   -0.8182   -0.9091
```

**Recall:**

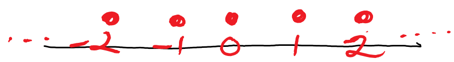
closed under addition (of vectors)

Recall:

Vector space = closed under addition (of vectors)
and multiplication (by scalars)
(plus a few other properties)

## Examples:

- $\mathbb{Z} = \{\text{integers}\}$




NOT a vector space
(closed under addition, but not multiplication)

- $\mathbb{Q} = \{\text{rationals } \frac{a}{b}\}$

addition ✓

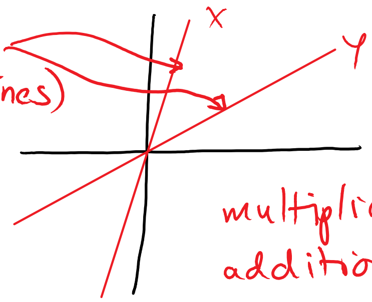
multiplication ✗

$(\pi \cdot 1 = \pi)$

scalar $\mathbb{Q}$ $\mathbb{Q}$

NOT a vector space

- this set (union of 2 lines)




multiplication ✓

addition ✗

NOT a vector space

## Examples: Vector spaces are everywhere!

① vectors

$\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

$\mathbb{C}^3$ is a vector space over $\mathbb{C}$ (over $\mathbb{R}$ too)

② matrices $\mathbb{R}^{m \times n}$

③ $\vec{0}$ sets

But these are NOT vector spaces:

$\{1\}$, $\{(1,0,0)\}$

$\{(0,0), (1,0)\}$

the interval $[0,1]$

④ function spaces

$\{$all functions from $S \overset{\shortparallel}{\mathbb{R}}$ to $\mathbb{R}\}$

$f \quad , g$

$$(f+g)(x) \equiv f(x)+g(x)$$

$$(\alpha g)(x) = \alpha \cdot g(x)$$

- polynomials of degree $\leq k = 3$
- differentiable functions $\mathbb{R} \to \mathbb{R}$

⑤ Subspaces!

**Exercise:** What are ALL subspaces of $\mathbb{R}^2$?

$\{0\}$ , lines through 0 $\{(x,y) : ax+by=0\}$ , $\mathbb{R}^2$ itself

NOT subspaces: other lines, curves

Important: Lines/planes/hyperplanes that **don't** go through the origin (0) are NOT subspaces!!

⑥ Examples over other fields

$\mathbb{F}_p$ = field of numbers mod $p$ (for a prime $p$)

$\mathbb{F}_2 = \{0, 1\}$

Bit strings of length $n$ form a vector space:

$n=3$: $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$
$(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$

addition is coordinate-wise, mod 2

$(0,0,1)+(0,1,1)=(0,1,0)$

subspace, eg.,

$\mathrm{Span}(\{(0,0,1),(1,0,1)\})$
$=\{(0,0,0),(0,0,1),(1,0,1),(1,0,0)\}$

Problems:

1. How many subspaces are there of $\mathbb{R}^2$?
   Answer: Infinitely many!
   (lines through the origin)
2. How many subspaces are there of $\mathbb{R}$?
   Answer: Two! $\{0\}$ and $\mathbb{R}$ itself.
3. How many subspaces are there of $\{0,1\}^2$?

   $\{(0,0)\}$, everything $\{0,1\}^2$

   $\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$

   $\{(0,0),(1,0)\}$, $\{(0,0),(1,1)\}$

   and that's it!

   (if a subspace contains two of the nonzero points, then it also includes their sum, which is the last nonzero point: $(1,0)+(0,1)+(1,1)=(0,0)$ means that any two sum to the third)

   $1+4+1=6$
4. How many subspaces are there of $\{0,1\}^n$?
   We'll answer this later! Definitely $<\infty$ though

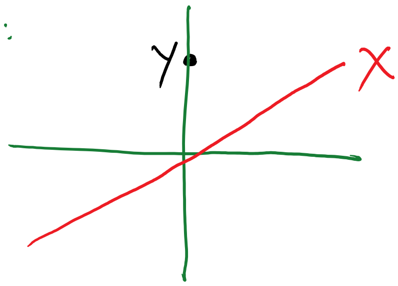
(8) The SUM of two subspaces is a subspace.

Definition: For two subsets $X$ and $Y$ in a vector space $V$, let space $V$, let

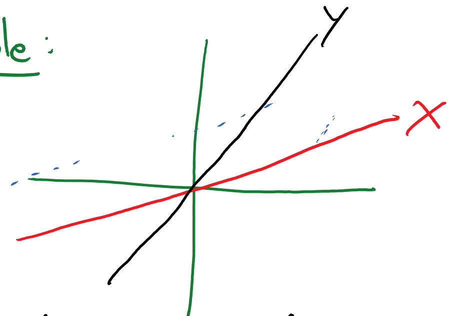
$$X + Y \doteq \{x + y \mid x \in X, y \in Y\}$$

(In English: all sums of a vector in $X$ and a vector in $Y$.)

Example:




Example:



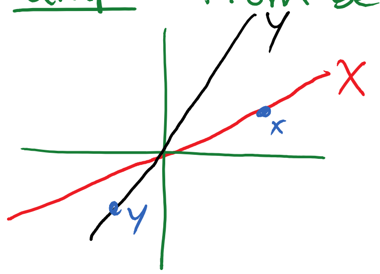

Claim 1: If $X$ and $Y$ are subspaces, then $X+Y$ is a subspace.

Claim 2: For subsets $S$ and $T$ of a vector space $V$,

$\text{Span}(S) + \text{Span}(T) = \text{Span}(S \cup T)$

$$\mathrm{Span}(S) + \mathrm{Span}(T) = \mathrm{Span}(S \,\dot{\cup}\, T)$$

**Example**: From before




$X = \mathrm{Span}(\{x\})$
$Y = \mathrm{Span}(\{y\})$
$\mathrm{Span}(\{x, y\}) = \mathbb{R}^2 = X + Y$

**Definition**: Affine subspace = translated subspace

i.e., a set $\vec{u} + V$ for a vector $\vec{u} \neq \vec{0}$ and subspace $V$.

③ The SPAN of any (finite or infinite) set of points $S$

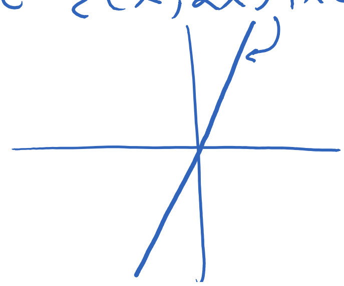
| |
|---|
| Span(S) is defined to be the set of all finite linear combinations of elements from $S$ |

i.e., all sums $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r$
for scalars $\alpha_j$ and $\vec{v}_j \in S$

By definition, this is closed under $+$, $\times$, and hence is a vector space.

**Examples**: What are:

- $\mathrm{Span}\{(1,2)\}$ = the line $\{(x, 2x) \mid x \in \mathbb{R}\}$
- $\mathrm{Span}\{(1,2), (-1,-2)\}$
  = the same line



= the same line

- Span $\{(0,1),(1,0)\}$ = the plane $\mathbb{R}^2$

- Span $\{1, x, x^2, x^3, \dots\}$ = all polynomials

  Note: The infinite sum $\sum_{j=0}^{\infty} x^j$ is not in this span.

- Span $\left\{\begin{array}{l}(1,0,0),\\(0,0,1),\\(1,0,1)\end{array}\right\}$ = the xz-plane in $\mathbb{R}^3$

  (since $\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$, it doesn't increase the span)

- Span $\left\{\begin{array}{l}(1,2,1,1,5),\\(-2,-4,0,4,-2),\\(1,2,2,4,9)\end{array}\right\}$

  - this is a subspace of $\mathbb{R}^5$ — it must be either a line, a plane, or a 3D hyperplane

Two approaches to find out:

① Add vectors one at a time

Span$\{(1,2,1,1,5)\}$ is a line.

Does $(-2,-4,0,4,-2)$ give something new, or is it already in that line?

— something new, since it is not a multiple of $(1,2,1,1,5)$

Does $(1,2,2,4,9)$ give something new, or is it already in the plane Span $\left\{\begin{array}{l}(1,2,1,1,5),\\(-2,-4,0,4,-2)\end{array}\right\}$?

It lies in the plane $\iff$ there is a solution to

$$(1,2,2,4,9) = (1,2,1,1,5)x + (-2,-4,0,4,-2)y$$

$$(1,2,2,4,9) = (1,2,1,1,5)x + (-2,-4,0,4,-2)y$$

$$\Longleftrightarrow \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 1 & 0 \\ 1 & 4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \\ 9 \end{pmatrix} \quad \text{has a solution}$$

There is a solution: $(x,y) = (2, \tfrac{1}{2})$

Therefore the span is the 2D plane

$$\text{Span}\left\{ \begin{matrix} (1,2,1,1,5), \\ (-2,-4,0,4,-2) \end{matrix} \right\}. \checkmark$$

(2) <u>Start with all the vectors, and try to simplify them</u>

This is what Gaussian elimination does:

$$\text{Let} \quad M = \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix} \end{array} \begin{array}{l} 2 \\ -1 \\ \end{array}$$

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \begin{array}{l} \\ -\tfrac{1}{2} \\ \end{array} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -1 \\ \\ \end{array}$$

$$\xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & & 0 & 0 \end{pmatrix}$$

<u>Note</u>: Adding a multiple of one row to another does not change the <u>span</u> of the rows.

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$$

$$= \text{Span}\{\vec{v}_1, \underbrace{\vec{v}_2 + \beta\vec{v}_1}_{= \vec{v}_2'}, \vec{v}_3, \ldots\}$$

since anything you can reach with a linear combination

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \cdots$$

can also be reached with a linear combination of the new vectors

$$(\alpha_1 - \beta\alpha_2)\vec{v}_1 + \alpha_2(\vec{v}_2 + \beta\vec{v}_1) + \alpha_3\vec{v}_3 + \cdots,$$

and vice versa.

$\Rightarrow$ The nonzero rows left over after Gaussian elimination (are a minimal set of vectors) that span the same set as the original rows.

**Exercise**: Prove (by contradiction) that Span(S) is the smallest vector space containing S.

**Claim**: Span(S) is the smallest vector space that contains all the points in S.

Proof:

- Let T be a vector space containing all of S.
- Let $\vec{v} \in \mathrm{Span}(S)$.

$\Rightarrow \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r$, with all $\vec{v}_i \in S$

$\Rightarrow$ all $\vec{v}_i \in T$

$\Rightarrow$ all $\alpha_i \vec{v}_i \in T$ (closure under mult.)

$\Rightarrow \vec{v} = \sum_i \alpha_i \vec{v}_i \in T$ (closure under addition)

$\Rightarrow \mathrm{Span}(S) \subseteq T$. ✓ $\square$

⑦ Polynomials $= \mathrm{Span}\{1, x, x^2, x^3, \ldots\}$

Continuous functions

Differentiable functions

Functions f with $f(1) = 0$, $f'(2) = 0$

## More examples of vector spaces

| Space | Closed under addition? | Closed under multiplication? | Vector space? |
|---|---|---|---|
| $V_1 = \left\{ (b_1, b_2, b_3) \in \mathbb{R}^3 \text{ s.t. } b_1 - 2b_2 + 3b_3 = 0 \right\}$ | ✓ | ✓ | ✓ yes |
| $V_2 = \left\{ (b_1, b_2, b_3) \in \mathbb{R}^3 \text{ s.t. } b_2 b_3 = 0 \right\}$ | ✗ | ✓ | ✗ no |
| $\{ b \in \mathbb{R}^3 \mid b_1 + b_2 = 1 \}$ | ✗ | ✗ | ✗ affine subspace $= \{(1,0)\} + V_1$ |
| $\mathrm{Span}(\{(1,1,0), (2,0,1)\})$ | ✓ | ✓ | ✓ |

| | | | | |
|---|---|---|---|---|
| $\mathrm{Span}(\{(1,1,0),(2,0,1)\})$ | ✓ | ✓ | ✓ | |
| {upper-triangular $m\times n$ matrices} | ✓ | ✓ | ✓ | |
| {diagonal $n\times n$ matrices} | ✓ | ✓ | ✓ | |
| $V_7 = \left\{ 10\times 10 \text{ matrices } A \text{ with } \mathrm{Trace}(A) \doteq \sum_{j=1}^{10} a_{jj} = 0 \right\}$ | ✓ | ✓ | ✓ | |
| $\{10\times 10$ matrices $A$ with $\mathrm{Tr}(A)=1\}$ | ✗ | ✗ | ✗ | <u>affine</u> subspace $= V_7 + \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & \end{pmatrix}\right\}$ |
| $\left\{3\times 3 \text{ matrices } A \text{ with } A\begin{pmatrix}1\\1\\1\end{pmatrix} = 0\right\}$ | ✓ | ✓ | ✓ | |
| {symmetric $n\times n$ matrices (i.e. $A=A^T$)} | ✓ | ✓ | ✓ | |
| {arithmetic progressions, i.e., sequences $(x_1, x_2, x_3, \dots)$ with $x_j - x_{j-1}$ constant} (e.g., $(0,2,4,6,8,10,\dots)$) | ✓ | ✓ | ✓ | |
| {differentiable functions $f:\mathbb{R}\to\mathbb{R}$ with $f'(2)=3$} | ✗ | ✗ | ✗ | affine subspace $= 3x + \{f \text{ with } f'(2)=0\}$ |
| $\{(0,0),(2,1),(4,2),(6,3),\dots,(94,47),(96,48)\} \subset (\mathbb{Z}/97\mathbb{Z})\times(\mathbb{Z}/97\mathbb{Z})$ (97 is prime) | ✗ | ✗ | ✗ | |
| {all matrices that commute with a given matrix $A$} | ✓ | ✓ | ✓ | |

*Proof.* To prove (4.1.1), demonstrate that the two closure properties **(A1)** and **(M1)** hold for $\mathcal{S} = \mathcal{X} + \mathcal{Y}$. To show **(A1)** is valid, observe that if $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, then $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{Y}$, and therefore $\mathbf{u} + \mathbf{v} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in \mathcal{S}$. To verify **(M1)**, observe that $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication so that $\alpha\mathbf{x}_1 \in \mathcal{X}$ and $\alpha\mathbf{y}_1 \in \mathcal{Y}$ for all $\alpha$, and consequently $\alpha\mathbf{u} = \alpha\mathbf{x}_1 + \alpha\mathbf{y}_1 \in \mathcal{S}$ for all $\alpha$. To prove (4.1.2), suppose $\mathcal{S}_X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\mathcal{S}_Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$, and write

$$\mathbf{z} \in span\,(\mathcal{S}_X \cup \mathcal{S}_Y) \Longleftrightarrow \mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i + \sum_{i=1}^{t} \beta_i \mathbf{y}_i = \mathbf{x} + \mathbf{y} \text{ with } \mathbf{x} \in \mathcal{X},\ \mathbf{y} \in \mathcal{Y}$$
$$\Longleftrightarrow \mathbf{z} \in \mathcal{X} + \mathcal{Y}. \quad \blacksquare$$

## Example 4.1.8

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law—sketch a picture for yourself.

## Exercises for section 4.1

**4.1.1.** Determine which of the following subsets of $\Re^n$ are in fact subspaces of $\Re^n$ $(n > 2)$.

(a) $\{\mathbf{x} \mid x_i \geq 0\}$, (b) $\{\mathbf{x} \mid x_1 = 0\}$, (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$,

(d) $\left\{\mathbf{x} \;\middle|\; \sum_{j=1}^{n} x_j = 0\right\}$, (e) $\left\{\mathbf{x} \;\middle|\; \sum_{j=1}^{n} x_j = 1\right\}$,

(f) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},$ where $\mathbf{A}_{m\times n} \neq \mathbf{0}$ and $\mathbf{b}_{m\times 1} \neq \mathbf{0}\}$.

**4.1.2.** Determine which of the following subsets of $\Re^{n\times n}$ are in fact subspaces of $\Re^{n\times n}$.

(a) The symmetric matrices. (b) The diagonal matrices.
(c) ~~The nonsingular matrices.~~ (d) ~~The singular matrices.~~
(e) The triangular matrices. (f) The upper-triangular matrices.
(g) All matrices that commute with a given matrix $\mathbf{A}$.
(h) All matrices such that $\mathbf{A}^2 = \mathbf{A}$.
(i) All matrices such that $trace\,(\mathbf{A}) = 0$.

**4.1.3.** If $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

**4.1.4.** Why must a real or complex nonzero vector space contain an infinite number of vectors?

**4.1.5.** Sketch a picture in $\Re^3$ of the subspace spanned by each of the following.

(a) $\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}2\\6\\4\end{pmatrix}, \begin{pmatrix}-3\\-9\\-6\end{pmatrix}\right\}$, (b) $\left\{\begin{pmatrix}-4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\5\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$,

(c) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

**4.1.6.** Which of the following are spanning sets for $\Re^3$ ?

(a) $\{(1\ 1\ 1)\}$ (b) $\{(1\ 0\ 0), (0\ 0\ 1)\}$,
(c) $\{(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (1\ 1\ 1)\}$,
(d) $\{(1\ 2\ 1), (2\ 0\ -1), (4\ 4\ 1)\}$,
(e) $\{(1\ 2\ 1), (2\ 0\ -1), (4\ 4\ 0)\}$.

**4.1.7.** For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why $span(\mathcal{M} \cup \mathcal{N}) = span(\mathcal{M}) + span(\mathcal{N})$.

skip

**4.1.8.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.
(a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$.
(b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

**4.1.9.** For $\mathbf{A} \in \Re^{m\times n}$ and $\mathcal{S} \subseteq \Re^{n\times 1}$, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. We refer to $\mathbf{A}(\mathcal{S})$ as the set of ***images*** of $\mathcal{S}$ under $\mathbf{A}$.
(a) If $\mathcal{S}$ is a subspace of $\Re^n$, prove $\mathbf{A}(\mathcal{S})$ is a subspace of $\Re^m$.
(b) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ spans $\mathcal{S}$, show $\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k$ spans $\mathbf{A}(\mathcal{S})$.

**4.1.10.** With the usual addition and multiplication, determine whether or not the following sets are vector spaces over the real numbers.
(a) $\Re$, (b) $\mathcal{C}$, (c) The rational numbers.

**4.1.11.** Let $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\}$ and $\mathcal{N} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space. Prove that $span(\mathcal{M}) = span(\mathcal{N})$ if and only if $\mathbf{v} \in span(\mathcal{M})$.

**4.1.12.** For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, prove that $span(\mathcal{S})$ is the intersection of all subspaces that contain $\mathcal{S}$. **Hint:** For $\mathcal{M} = \bigcap_{\mathcal{S}\subseteq\mathcal{V}} \mathcal{V}$, prove that $span(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span(\mathcal{S})$.

**Nullspace is a line** $\begin{bmatrix}1&0&1\\5&4&9\\2&4&6\end{bmatrix}\begin{bmatrix}c & c & -c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

The nullspace of $B$ is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix}1&0&1\\5&4&9\\2&4&6\end{bmatrix}\begin{bmatrix}c & c & -c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The nullspace of $B$ is the line of all points $x = c$, $y = c$, $z = -c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax = b$, to find $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$: all attainable right-hand sides $b$ and all solutions to $Ax = 0$.

The vectors $b$ are in the column space and the vectors $x$ are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to $A$—the column space of $A$, the nullspace of $A$, and their two perpendicular spaces.

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## Problem Set 2.1

**1.** Construct a subset of the $x$-$y$ plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

*Hint*: Starting with $u$ and $v$, add and subtract for (a). Try $cu$ and $cv$ for (b).

**2.** Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors $(b_1, b_2, b_3)$ with first component $b_1 = 0$.

(b) The plane of vectors $b$ with $b_1 = 1$.

(c) The vectors $b$ with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).

(d) All combinations of two given vectors $(1,1,0)$ and $(2,0,1)$.

(e) The plane of vectors $(b_1, b_2, b_3)$ that satisfy $b_3 - b_2 + 3b_1 = 0$.

**3.** Describe the column space and the nullspace of the matrices

$$A = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0&0&3\\1&2&3\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

**4.** What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

**5.** Addition and scalar multiplication are required to satisfy these eight rules:

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all $x$.
4. For each $x$ there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3,1)+(5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers, with $x+y$ and $cx$ redefined to equal the usual $xy$ and $x^c$, is a vector space. What is the "zero vector"?

(c) Suppose $(x_1,x_2)+(y_1,y_2)$ is defined to be $(x_1+y_2,x_2+y_1)$. With the usual $cx=(cx_1,cx_2)$, which of the eight conditions are not satisfied?

**6.** Let $\mathbf{P}$ be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

**7.** Which of the following are subspaces of $\mathbf{R}^\infty$?

(a) All sequences like $(1,0,1,0,\dots)$ that include infinitely many zeros.

(b) All sequences $(x1,x2,\dots)$ with $x_j=0$ from some point onward.

(c) All decreasing sequences: $x_{j+1}\le x_j$ for each $j$.

(d) All convergent sequences: the $x_j$ have a limit as $j\to\infty$.

(e) All arithmetic progressions: $x_{j+1}-x_j$ is the same for all $j$.

(f) All geometric progressions $(x_1,kx_1,k^2x_1,\dots)$ allowing all $k$ and $x_1$.

**8.** Which of the following descriptions are correct? The solutions $x$ of

$$Ax=\begin{bmatrix}1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

form

(a) a plane.

(b) a line.

(c) a point.

(d) a subspace.



(e) the nullspace of $A$.

(e) the nullspace of $A$.

(f) the column space of $A$.

**9.** Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

**10.** The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space **M** of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing $A$?

**11.** (a) Describe a subspace of **M** that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of **M** contains $A$ and $B$, must it contain $I$?

(c) Describe a subspace of **M** that contains no nonzero diagonal matrices.

**12.** The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in the vector space **F** of all real functions. The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ____. Which rule is broken if multiplying $f(x)$ by $c$ gives the function $f(cx)$?

**13.** If the sum of the "vectors" $f(x)$ and $g(x)$ in **F** is defined to be $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

**14.** Describe the smallest subspace of the 2 by 2 matrix space **M** that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

**15.** Let **P** be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0,0,0)$ is not in **P**! Find two vectors in **P** and check that their sum is not in **P**.

**16.** $\mathbf{P}_0$ is the plane through $(0,0,0)$ parallel to the plane **P** in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

**17.** The four types of subspaces of $\mathbf{R}^3$ are planes, lines, $R^3$ itself, or **Z** containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe the five types of subspaces of $\mathbf{R}^4$.

**18.** (a) The intersection of two planes through $(0,0,0)$ is probably a ____ but it could be a ____. It can't be the zero vector **Z**!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____ but it could be a ____.

(c) If **S** and **T** are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S}\cap\mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on* $x+y$ *and* $cx$.

**19.** Suppose **P** is a plane through $(0,0,0)$ and **L** is a line through $(0,0,0)$. The smallest vector space containing both **P** and **L** is either ____ or ____.

**20.** True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in **M** (with $A^{\mathrm{T}} = -A$) form a subspace.

(b) The unsymmetric matrices in **M** (with $A^{\mathrm{T}} \neq A$) form a subspace.

(c) The matrices that have $(1,1,1)$ in their nullspace form a subspace.

**Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.**

**21.** Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

**22.** For which right-hand sides (find a condition on $b_1, b_2, b_3$) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

**23.** Adding row 1 of $A$ to row 2 produces $B$. Adding column 1 to column 2 produces $C$. A combination of the columns of ____ is also a combination of the columns of $A$. Which two matrices have the same column ____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

**24.** For which vectors $(b_1, b_2, b_3)$ do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

**25.** (Recommended) If we add an extra column $b$ to a matrix $A$, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including $b$?

**26.** The columns of $AB$ are combinations of the columns of $A$. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A*. Give an example where the column spaces of $A$ and $AB$ are not equal.



**27.** If $A$ is any 8 by 8 invertible matrix, then its column space is ____. Why?

**27.** If $A$ is any 8 by 8 invertible matrix, then its column space is ____. Why?

**28.** True or false (with a counterexample if false)?

(a) The vectors $b$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then $A$ is the zero matrix.

(c) The column space of $2A$ equals the column space of $A$.

(d) The column space of $A - I$ equals the column space of $A$.

**29.** Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

**30.** If the 9 by 12 system $Ax = b$ is solvable for every $b$, then $\boldsymbol{C}(A) =$ ____.

**31.** Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

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## 2.2 Solving $Ax = 0$ and $Ax = b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$ and it was $x = -A^{-1}b$. That solution was found by elimination (not by computing $A^{-1}$). A rectangular matrix brings new possibilities—$U$ may not have a full set of pivots. This section goes onward from $U$ to a reduced form $R$—**the simplest matrix that elimination can give**. $R$ reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by $A^{-1}$). The column space is the whole space ($Ax = b$ has a solution for every $b$). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector $x_n$ in the nullspace can be added to a particular solution $x_p$. The solutions to all linear equations have this form, $x = x_p + x_n$:

   **Complete solution** $\quad Ax_p = b$ **and** $Ax_n = 0$ **produce** $A(x_p + x_n) = b$.

2. When the column space doesn't contain every $b$ in $\mathbf{R}^m$, we need the conditions on $b$ that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for $b$ to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* $x$. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 + (\text{any } x)$.

More important examples:

SUBSPACES OF A MATRIX

Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ real-valued matrix.

- Range($A$)

**Observe:** $\vec{b} \in \text{Range}(A) \iff A\vec{x} = \vec{b}$ has a solution

- Range($A^T$)

- Kernel($A$)