Lecture 6: Subspaces of a matrix

Thursday, September 10, 2015 9:30 AM

Admin: Reading: Start Ch. 4 of Meyer

Definition: For an $m \times n$ matrix A,

- The nullspace of A is

  $$N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}.$$

  (everything A sends to $\vec{0}$)

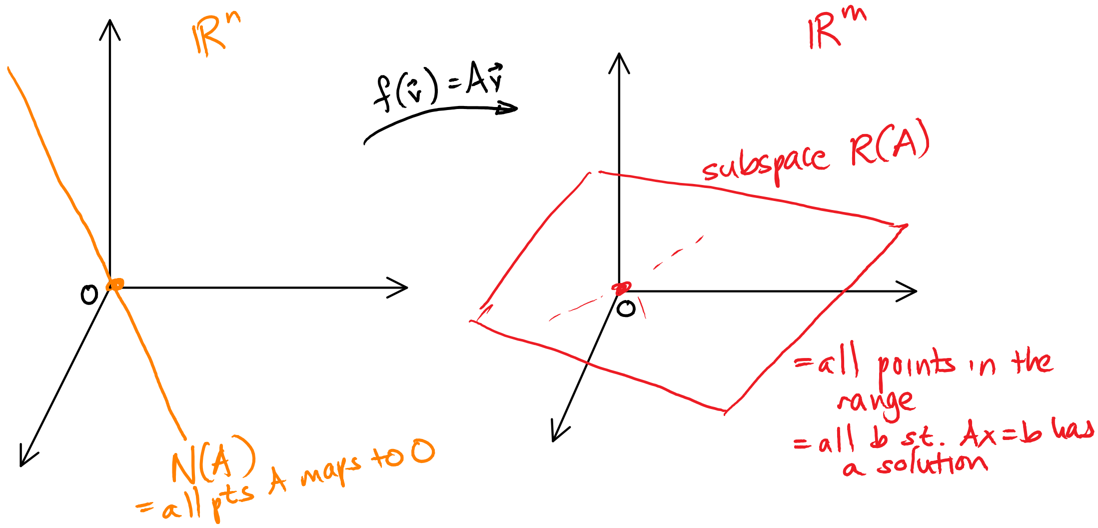
- The range (or column space) of A is

  $$R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
  $$= \text{Span(columns of } A)$$

  (everything reachable by A)

- The rowspace of A is $R(A^T)$

  (span of the rows of A)



Observe: $\vec{b} \in \text{Range}(A) \iff A\vec{x} = \vec{b}$ has a solution

Example: Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and let

$$A = \vec{u}\,\vec{v}^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}(3 \quad 4) = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}.$$

What are $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$?

Answer: Yes, we could apply Gaussian elimination.
But we can also just observe that for any vector $\vec{x}$,

$$A\vec{x} = \vec{u}\,\vec{v}^T\vec{x} = \underbrace{(\vec{v}^T\vec{x})}_{\in \mathbb{R}} \cdot \vec{u}$$

$\Rightarrow R(A) = \text{Span}(\{\vec{u}\})$, a line.

Similarly, $R(A^T) = \text{Span}(\{\vec{v}\})$, a line.

Also $A\vec{x} = 0 \Leftrightarrow \underbrace{v^T x}_{3x_1 + 4x_2} = 0$

$\Leftrightarrow x \propto (-4, 3)$

$\Rightarrow N(A) = \text{Span}(\{(-4,3)\})$, the line perpendicular to u

Similarly $N(A^T) = \text{Span}(\{(-2,1)\})$. ✓

NULLSPACE $\{x \in \mathbb{R}^n \mid Ax = 0\}$

i.e., the set of solutions to the homogeneous equations specified by A

Claim 1: This is a subspace!
(The solutions to a set of homogeneous equations form a subspace.)

Proof:

- closure under multiplication:
  If x is a solution and α any scalar, then
  $$A(\alpha x) = \alpha A x = \alpha \cdot 0 = 0$$
  $\Rightarrow \alpha x$ is a solution ✓
- closure under addition:
  If x and y are both solutions, then
  $$A(x+y) = Ax + Ay = 0 + 0 = 0$$
  $\Rightarrow x+y$ is a solution ✓

$\square$

Problem:
Let $M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$

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What is $N(M)$?

Answer: Solving for $N(M) \Leftrightarrow$ Solving $M\vec{x} = 0$

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_1 = -2x_2 + 2x_4 - x_5$

$\quad x_3 = -3x_4 - 4x_5$

$\Rightarrow$ General solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{N(M) = \text{Span}\left(\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}\right)} \checkmark$$

Observe: Row operations don't change the nullspace.

Claim 2: Consider the equations $Ax = b$, with $b \neq 0$.
Let y be a solution ($Ay = b$). Then

$$\{x \mid Ax = b\} = y + \{x \mid Ax = 0\}$$
$$= y + N(A)$$

Thus, the solutions to a set of **non-homogeneous** equations form an **affine subspace**.

form an affine subspace.

(In English: The general solution of the nonhomogeneous system is given by a particular solution (y) plus the general solution of the associated homogeneous system. [We claimed this in Lecture 2, but didn't prove it.])

Proof: Let $V_b = \{x \mid Ax = b\}$.

To show equality, we need to verify $V_b \subseteq y + N(A)$ and $V_b \supseteq y + N(A)$.

- $V_b \supseteq y + N(A)$:
  Indeed, if $z \in N(A)$, then $A(y+z) = Ay + Az = b + 0 = b$
  $\Rightarrow y + z \in V_b$ ✓
- $V_b \subseteq y + N(A)$:
  Indeed, if $x \in V_b$, then $Ax = b$, so
  $A(y-b) = Ay - Ax = b - b = 0$
  $\Rightarrow y - b \in N(A)$ ✓ $\square$

Corollary: The equations $Ax = b$ have infinitely many solutions if and only if $N(A) = \{0\}$.

RANGE $\{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

Claim 3: For $S \subseteq \mathbb{R}^n$, let

$$A(S) = \{A\vec{x} \mid \vec{x} \in S\}$$

(map the set S forward by applying A)

1. If S is a subspace, so is $A(S)$.
2. If $s_1, \dots, s_k$ span S, then $As_1, \dots, As_k$ span $A(S)$.

Proof:

1) Closure under addition:

Let $y, z \in A(S)$. Is $y+z \in A(S)$?
Yes! $y = Ax$, $z = Ax'$
$y + z = A(x + x') \in A(S)$ ✓

Closure under multiplication is similar ✓

2) Any vector $y \in A(S)$ can be written $y = Ax$ for some $x \in S$.
$s_1, \dots, s_k$ span $S \Rightarrow x = \sum_{j=1}^{k} \alpha_j s_j$ for some scalars $\alpha_1, \dots, \alpha_k$.
$\Rightarrow$ by linearity, $y = Ax = \sum_{j=1}^{k} \alpha_j (As_j)$
$\Rightarrow y \in \mathrm{Span}(\{As_1, \dots, As_k\})$ ✓ $\square$

<u>Observe</u>: ① $b \in R(A) \iff Ax = b$ has a solution.
② $R(A) = A(\mathbb{R}^n)$
$= \mathrm{Span}(\text{columns of } A)$

Why?
$\mathbb{R}^n$ is spanned by $(1,0,0,\dots,0), \dots, (0,0,\dots,0,1)$
By the exercise, $A(\mathbb{R}^n)$ is spanned by
$A\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = 1^{st}$ column, $\dots$, $A\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix} =$ last column of A ✓

Thus the range is also known as the column space of A.

<u>Observe</u>: ③ $A_{n\times n}$ is invertible (A^{-1} exists)
$\iff$ every point $b \in \mathbb{R}^m$ has a preimage x, $Ax = b$
$\iff R(A) = \mathbb{R}^n$ (everything)

<u>ROW SPACE</u> $R(A^T) = \mathrm{Span}(\text{rows of } A)$

<u>Problem</u>:
Let $M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$ (as before)

What are the row and column spaces of M?

<u>Answer</u>:

$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \end{pmatrix}$

Answer.

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe: The row operations of Gaussian elimination do not change the row space.

(Because $\mathrm{Span}(v_1, v_2, v_3, \ldots, v_k) = \mathrm{Span}(v_1, v_2 + \alpha v_1, v_3, \ldots, v_k)$.)

$$\Rightarrow \underbrace{R(M^T)}_{\text{row space}} = \mathrm{Span}\{(1,2,0,-2,1), (0,0,1,3,4)\}. \checkmark$$

How can we compute $R(M)$?

① Apply Gaussian elimination to M^T

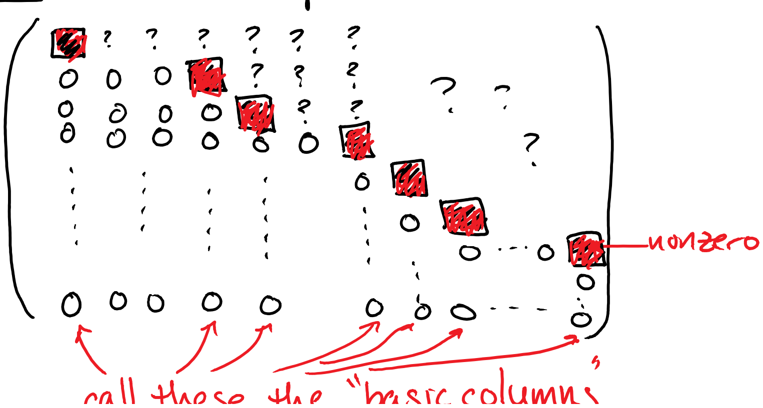
$$M^T = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & 0 & 2 \\ 1 & 4 & 4 \\ 5 & -2 & 9 \end{pmatrix} \begin{matrix} \\ -2 \\ -1 \\ -1 \\ -5 \end{matrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \\ 0 & 8 & 4 \end{pmatrix} \begin{matrix} \\ \\ \\ -3 \\ -4 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R(M) = \mathrm{Span}(\{(1,-2,1), (0,2,1)\})$$

② But if we have already applied G.E. to M, there is no need to apply it again to M^T.

Observe: The row operations of G.E. finish with a matrix like



Claim: Every column is a linear combination of the basic columns.

Proof: Starting with a non-basic column, use the last basic column to cancel out its last coordinate, etc., up to using the first basic column to cancel out the first coordinate. $\square$

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix} \xrightarrow{G.E.} G = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 5 \\ 0 & 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

basic columns

e.g. $(1,3,0) = 3(1,1,0) - 2(1,0,0)$ ✓

The **same** linear combination of columns works for M:

$$\underset{\substack{\text{column } 4 \\ M\vec{e}_4}}{(1,4,4)} = 3\underset{\substack{\text{column } 3 \\ M\vec{e}_3}}{(1,-2,1)} - 2\underset{\substack{\text{column } 1 \\ M\vec{e}_1}}{(1,-2,1)}$$

Why?

$M = QG$, where Q is the matrix that implements the row operations in Gaussian Elim.

$\Rightarrow$ If $\underset{j\text{th col}}{Ge_j} = \sum_k \alpha_k \underset{k\text{th column}}{Ge_k}$,

then $QGe_j = Q\left(\sum_k \alpha_k Ge_k\right)$

$$\underset{\substack{\| \\ Me_j}}{QGe_j} = \underset{\substack{\| \\ \sum_k \alpha_k Me_k}}{Q\left(\sum_k \alpha_k Ge_k\right)}$$

$\Rightarrow \text{Range}(M) = \text{Span}\left(\{(1,-2,1),(1,0,2)\}\right)$

In general:

After Gaussian elimination, identify the basic columns.

$R(M) = \text{Span}$(those same basic columns in M).

Exercise: Prove

① $R(AB) \subseteq R(A)$

(Multiplying on the right can only <u>reduce</u> the range.)

$R(AB) = R(A)$ if B is invertible.

② $N(BA) \supseteq N(A)$

(Multiplying on the left can only increase the nullspace.)

$N(BA) = N(A)$ if B is invertible

Easy example: Take $B = 0$, so $AB = 0$, $BA = 0$

$R(AB) = \{0\} \subseteq R(A)$, $N(BA) = \mathbb{R}^n \supseteq N(A)$ ✓

Note: These are all sensitive to perturbations/numerical errors.

E.g.

	Range	Nullspace
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	x-axis $\{(x,0,0) \mid x \in \mathbb{R}\}$	yz-plane $\{(0,y,z) \mid y,z \in \mathbb{R}^2\}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$, $\varepsilon \neq 0$	$\mathbb{R}^3$	$\{0\}$

Example: What is the nullspace of

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} ?$$

Answer:

Intuition: Recall that A is the matrix we got from discretizing the second derivative operator

$$f''(t) = \lim_{\delta t \to 0} \frac{1}{(\delta t)^2}\left(f(t+\delta t) - 2f(t) + f(t-\delta t)\right).$$

The functions with second derivative 0 are exactly constant functions $f(t) = \text{constant}$.

$\Rightarrow$ We expect the nullspace of A to be the set of vectors with all-equal coordinates.

Claim: $N(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$

$= \text{set of all vectors with all-equal coordinates.}$

Proof:

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \text{sum of } a_{ij} \text{ across row } 1 \\ \vdots \\ \text{sum across row } n \end{pmatrix} = 0$$

$\Rightarrow (1, 1, \dots, 1) \in N(A)$

$\Rightarrow \operatorname{Span}(\{(1, \dots, 1)\}) \subseteq N(A).$

But are there other vectors in $N(A)$?

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{array}{l} +1 \\ \\ \\ \\ \\ \\ \\ \end{array} \to \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{array}{l} \\ +1 \\ \\ \\ \\ \\ \\ \end{array}$$

$$\to \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 0 & -1 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{array}{l} \\ \\ +1 \\ \\ \\ \\ \\ \end{array}$$

$$\vdots$$

$$\to B = \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 0 & -1 & 1 & & & \\ & & 0 & -1 & 1 & & \\ & & & & \ddots & & \\ & 0 & & & 0 & -1 & 1 \\ & & & & & 1 & -1 \end{pmatrix}$$

$A\vec{x} = 0 \iff B\vec{x} = 0$

$\iff x_1 = x_2$ (first row)

$ x_2 = x_3$ (2^{nd} row)

$\vdots$

$x_{n-1} = x_n$

$x_2 = x_3$ (2nd row)

$\vdots$

$x_{n-1} = x_n$

$\iff \vec{x} \in \text{Span}\{(1, 1, 1, \ldots, 1)\}$

<u>Question</u>: What is $R(A)$? (Note $A = A^T$.)