Lecture 8 : Linear independence, bases and dimension

Thursday, September 17, 2015 9:30 AM

Admin: Reading: Ch. 4.3–4.4

Today: Linear independence, bases and dimension

Recall:

Span (a set of vectors) = smallest subspace that contains them all

$$= \left\{ \begin{array}{c} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \end{array} \right\}$$

Example:

$$\mathbb{R}^3 = \mathrm{Span}\{\underset{\vec{e}_1}{(1,0,0)}, \underset{\vec{e}_2}{(0,1,0)}, \underset{\vec{e}_3}{(0,0,1)}\}$$

$$= \mathrm{Span}\{\vec{e}_1, \underbrace{\vec{e}_1+\vec{e}_2}_{(1,1,0)}, \underbrace{\vec{e}_1+\vec{e}_2+\vec{e}_3}_{(1,1,1)}, \vec{e}_1-\vec{e}_2, \vec{e}_2-\vec{e}_3\}$$

$$= \mathrm{Span}(\mathbb{R}^3)$$

Example:

Span (a finite set of vectors $v_1, \ldots, v_n$)

$$= R\left(\text{matrix}\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\right) \quad \text{range/column space}$$

$$\underbrace{\text{matrix}\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{= v_1 e_1^T + v_2 e_2^T + \cdots + v_n e_n^T}$$

Goal for today: Given a subspace, find ~~the~~ a smallest possible spanning set for it.

(smaller → simpler, easier to work with/understand)

Outline: Linear independence

Basis

Dimension

Rank, Rank-nullity theorem

LINEAR INDEPENDENCE

Definitions:

- A vector v is linearly independent of a set S if
$$v \notin \mathrm{Span}(S)$$

(If $v \in \mathrm{Span}(S)$, we say "v is linearly dependent on S.")

- A set of vectors is linearly independent if no vector can be expressed as a linear combination of the others (i.e., every vector is linearly independent of the others).

Example: Assume $S = \{v_1, v_2, v_3\}$ is linearly independent.

$$\Rightarrow v_1 \notin \mathrm{Span}(v_2, v_3)$$
$$v_2 \notin \mathrm{Span}(v_1, v_3)$$
$$v_3 \notin \mathrm{Span}(v_1, v_2)$$

$\Rightarrow$ Knowing $f(v_2)$, $f(v_3)$ tells us nothing about $f(v_1)$, etc.

Equivalent definition: A set $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent if the only solution to
$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$
is trivial: $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Why are these definitions equivalent?

- If $v_1 \in \mathrm{Span}(v_2, v_3)$, say $v_1 = \alpha_2 v_2 + \alpha_3 v_3$, then
$$-v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0.$$
- If $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ with $\alpha_j \neq 0$, then
$$v_j = -\frac{1}{\alpha_j}\Big(\sum_{i \neq j} \alpha_i v_i\Big) \in \mathrm{Span}(\{v_i \mid i \neq j\}).$$

Observe:

$S = \{\vec{v}_1, \dots, \vec{v}_n\}$ $\Longleftrightarrow$ nullspace of $\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \end{pmatrix}$ is $\{\vec{0}\}$

$S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent $\iff$ nullspace of $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is $\{\vec{0}\}$.

To check if a set is linearly independent, compute the nullspace.

BASIS AND DIMENSION

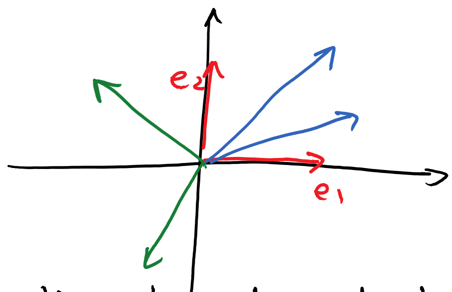
Definition: A **basis** for a vector space V is a **linearly independent** set that **spans** V.

(**Theorem**: Any two bases for the same space must have the **same size**.)

Definition: The **dimension** of a vector space V = # of vectors in a basis for V.

EXAMPLES: Different bases for the same space

- $V = \mathbb{R}^2$



$e_1 = (1,0)$
$e_2 = (0,1)$

any two linearly independent vectors form a basis
(because 2 lin. ind. vectors must span the plane)

- $\mathbb{R}^n$ dimension n

bases: • standard basis

linearly indep. ✓
span ✓

$$\{\underset{\vec{e}_1}{(1,0,0,\dots,0)}, \underset{\vec{e}_2}{(0,1,0,\dots,0)}, \dots, \underset{\vec{e}_n}{(0,0,\dots,0,1)}\}$$

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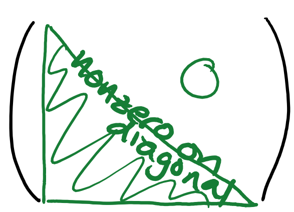


the columns are a basis
linearly independent ✓
span $\mathbb{R}^n$

- Haar basis (if n is a power of 2)

$$\{(1,1,1,1), (1,1,-1,-1), (1,-1,0,0), (0,0,1,-1)\}$$

$$\vec{x} = (x_1, x_2, x_3, x_4) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$$

$$= a(1,1,1,1) + b(1,1,-1,-1) + c(1,-1,0,0) + d(0,0,1,-1)$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow a = \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad c = \frac{x_1 - x_2}{2}$$

$$b = \frac{x_1 + x_2 - x_3 - x_4}{4}, \quad d = \frac{x_3 - x_4}{2}$$

- Any linearly independent set S is a basis for Span(S).

dimension $= |S|$

- $V = \{$polynomials of degree $\le n\}$
  - $\{1, x, x^2, \dots, x^n\}$ form a basis
  - Hermite polynomials $\{1, x, x^2-1, x^3-3x, x^4-6x^2+3, \dots, (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})\}$ is another

dimension $= n+1$

- $\mathbb{C}^n$ n dimensions over $\mathbb{C}$
  - standard basis $e_1, \dots, e_n$
  - Fourier basis $v_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \omega^{jk} e_k$

where $\omega = e^{2\pi i/n}$

Exercise: Why is $\{v_j\}$ linearly independent?

Why care?

Why care?

1. Makes all (finite-dimensional, real) vector spaces look just like $\mathbb{R}^n$

   e.g. $a + bx + cx^2 \longleftrightarrow (a, b, c) \in \mathbb{R}^3$

   for the polynomial basis $\{1, x, x^2\}$

2. Specifying a linear transformation on a basis gives its values everywhere.

   $$f(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \sum_j \alpha_j f(v_j)$$

3. Dimension measures the "size" of a vector space...

Theorem 1: Any two bases for the same space must have the **same # of elements**. ("Dimension makes sense")

Theorem 2: If $\mathcal{U}$ is a subspace of V and $\mathcal{U} \neq V$, then

$$\dim(\mathcal{U}) < \dim(V).$$

Proof of Theorem 1:

Let $S = \{s_1, \ldots, s_m\}$, $T = \{t_1, \ldots, t_n\}$ be two bases for a vector space V.

Assume $m < n$.

Goal: Show T is linearly dependent.

Let

$$B = \begin{pmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{pmatrix}$$

be the matrix whose columns are the vectors t_j expanded in the basis S. (Thus $t_j = \sum_{i=1}^{m} B_{ij} s_i$.)

B is an $m \times n$ matrix.

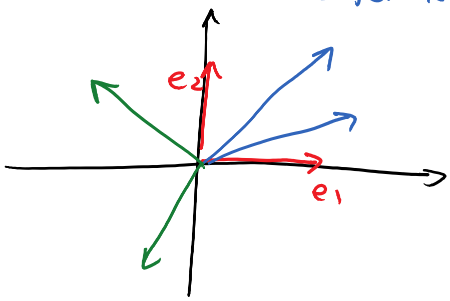
Recall **Lemma**: If A is an $m \times n$ matrix with $m < n$, then $N(A) \neq \{0\}$.

$\Rightarrow N(B) \neq \{0\}$

$\Rightarrow$ its columns, the vectors in T, are linearly dependent. ✓ $\square$

Corollary: If $\dim(V) = n$, then any linearly independent set $S \subseteq V$ with n elements is a basis for V.

We saw this before for $\mathbb{R}^2$:



Proof idea: If $\mathrm{Span}(S) \neq V$, then there is $v \in V$, $v \notin \mathrm{Span}(S)$.
Let $T = S \cup \{v\}$, with $n+1$ elts.
By the above argument, T is linearly dependent, a contradiction. $\square$

Theorem 2: If U is a subspace of V and $U \neq V$, then
$$\dim(U) < \dim(V).$$

Proof: Assume $\dim(U) = \dim(V) = n$.
Let S be a basis for U. By the corollary, S is a basis for V, too: $U = V$. Contradiction. $\square$

Another way to think about it is that a basis is a minimal spanning set is a maximal linearly independent set.
(Removing anything breaks the spanning property, and adding anything breaks linear independence.)
Use the proof of Theorem 1 to show this formally.

EXAMPLES

Space	Dimension	Basis
• 2×2 matrices	4	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$
• 2×2 symmetric matrices $A = A^T = \begin{pmatrix}a&b\\b&c\end{pmatrix}$	3	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$

- 2×2 antisymmetric matrices $\quad 1 \quad \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

  $A = -A^T = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$

- polynomials in x of degree ≤ 2 $\quad 3 \quad \{1, x, x^2\}$

  $a + bx + cx^2$

- polynomials in x and y with total degree ≤ 2 $\quad 6 \quad \{1, x, x^2, y, y^2, xy\}$

  $a + bx + cx^2 + dy + ey^2 + fxy$

- polynomials in x $\quad \infty \quad \{1, x, x^2, x^3, \ldots\}$

- $N((1\ 1\ 1\ \cdots\ 1))$ $\quad n-1 \quad \{\vec{e}_1 - \vec{e}_2, \vec{e}_1 - \vec{e}_3, \ldots, \vec{e}_1 - \vec{e}_n\}$

  $= \{\vec{x} \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 0\}$

- functions $f: \mathbb{R} \to \mathbb{R}$ $\quad \infty\,!$

- functions $f: \{0,1\}^2 \longrightarrow \mathbb{R}$ $\quad 4$

- functions $f: \{0,1\}^2 \to \mathbb{R}$ that are symmetric: $f(x,y) = f(y,x)$ $\quad 3$

  $f(0,0),\ f(0,1) = f(1,0),\ f(1,1)$

LINEAR TRANSFORMATIONS

$\updownarrow$

MATRICES

$f: \mathcal{U} \longrightarrow V$ linear transformation

basis $\{u_1, \ldots, u_n\}$ of $\mathcal{U}$; basis $\{v_1, \ldots, v_m\}$ of V

Since $f\left(\sum_j \alpha_j \vec{u}_j\right) = \sum_j \alpha_j f(\vec{u}_j)$,

f is determined by the values $f(\vec{u}_1), \ldots, f(\vec{u}_n)$.

Expand $f(\vec{u}_j) = \sum_i a_{ij} \vec{v}_i$

$$\Rightarrow A = \begin{array}{c|ccccc} & 1 & \cdots & j & \cdots & n \\ \hline 1 & & & & & \\ \vdots & & & & & \\ i & & & a_{ij} & & \\ \vdots & & & & & \\ m & & & & & \end{array}$$ determines

More precisely, then,

Matrix + bases $\longrightarrow$ Linear transformation

Linear transformation + bases $\longrightarrow$ Matrix

Remark: If you use different bases, you'll get different matrices, for the **same** linear transformation.

EXAMPLES

Example 1: Matrix transpose

$\mathcal{U} = \mathcal{V} = \{2 \times 2 \text{ matrices}\}$

$f : \mathcal{U} \to \mathcal{V}, \quad f(A) = A^T$

In the basis $\left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$

f corresponds to
$$\begin{array}{c|cccc} & \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) & \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) & \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) & \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \\ \hline \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) & 1 & 0 & 0 & 0 \\ \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) & 0 & 0 & 1 & 0 \\ \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) & 0 & 1 & 0 & 0 \\ \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) & 0 & 0 & 0 & 1 \end{array}$$

Observe: Order matters!

In the basis $\left\{ \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$,

f corresponds to $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

$\{ (1\ 0 \quad (1\ 0 \quad (0\ 1 \quad (0\ 1 \quad \}$

In the basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$,

f corresponds to $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$.

Remark: A basis in which a linear transformation acts diagonally is called an "eigenbasis". These are useful...

Example 2: $f(x,y,z) = (x,y)$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

- using the standard basis for both

$$[f] = \begin{array}{c} \\ \vec{e}_1 \\ \vec{e}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

- using $\{(1,1), (1,-1)\}$ as the basis for $\mathbb{R}^2$

$$[f] = \begin{array}{c} \\ \binom{1}{1} \\ \binom{1}{-1} \end{array} \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{array}$$

since $f(\vec{e}_1) = \binom{1}{0} = \frac{1}{2}\binom{1}{1} + \frac{1}{2}\binom{1}{-1}$

$f(\vec{e}_2) = \binom{0}{1} = \frac{1}{2}\binom{1}{1} - \frac{1}{2}\binom{1}{-1}$

$f(\vec{e}_3) = 0$

CHANGING BASIS

Notation: $[f]_{B_U, B_V}$

denotes the matrix representation of $f : U \to V$ in the respective bases B_U, B_V.

Example: $g(x,y) = (x,y)$

the identity $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

In any basis $\{\vec{u}_1, \vec{u}_2\}$,

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But for $\vec{u}_1 = (1,1)$, $\vec{u}_2 = (1,-1)$

$$\vec{e}_1 = \tfrac{1}{2}(\vec{u}_1 + \vec{u}_2), \quad \vec{e}_2 = \tfrac{1}{2}(\vec{u}_1 - \vec{u}_2)$$

$$[g]_{\{e_1,e_2\} \to \{\vec{u}_1, u_2\}} = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{array}$$

For $f(x,y,z) = (x,y)$,

$$[f]_{\{e_1,e_2,e_3\},\{\vec{e}_1,\vec{e}_2\}} = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$[f]_{\{e_1,e_2,e_3\},\{u_1,u_2\}} = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\mathcal{U} \text{ basis } B_\mathcal{U} \xrightarrow{f} \mathcal{V} \text{ basis } B_\mathcal{V} \xrightarrow{\text{identity}} \mathcal{V} \text{ basis } B'_\mathcal{V}$$

$$[f]_{B_\mathcal{U}, B'_\mathcal{V}} = [\text{identity}]_{B_\mathcal{V}, B'_\mathcal{V}} \, [f]_{B_\mathcal{U}, B_\mathcal{V}}$$

You can similarly change the basis for $\mathcal{U}$ by computing $[\text{identity}]_{B'_\mathcal{U}, B_\mathcal{U}}$:

$$\mathcal{U} \text{ basis } B'_\mathcal{U} \xrightarrow{\text{identity}} \mathcal{U} \text{ basis } B_\mathcal{U} \xrightarrow{f} \mathcal{V} \text{ basis } B_\mathcal{V}$$

$$[f]_{B'_\mathcal{U} B_\mathcal{V}} = [f]_{B_\mathcal{U}, B_\mathcal{V}} \, [\text{identity}]_{B'_\mathcal{U}, B_\mathcal{U}}$$

HARDER EXAMPLES

① Polynomial differentiation $\quad f(p) = \frac{d}{dx} p$

$$f : \underbrace{\left\{ \begin{array}{c} \text{polynomials in } x \\ \text{of degree} \le 4 \end{array} \right\}}_{\substack{\| \\ \mathcal{U}}} \longrightarrow \underbrace{\left\{ \begin{array}{c} \text{polynomials} \\ \text{of degree} \le 3 \end{array} \right\}}_{\substack{\| \\ \mathcal{V}}}$$

5-dim $\qquad\qquad$ 4-dim.

$B_u = \{1, x, x^2, x^3, x^4\}$ $\qquad$ $B_v = \{1, x, x^2, x^3\}$

$$[f]_{B_u, B_v} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

since $f(1) = 0$, $f(x) = 1$, $f(x^2) = 2x$, $f(x^3) = 3x^2$, $f(x^4) = 4x^3$

Different bases will give different matrices.

$B_u' = \{1, x, x^2 - 1, x^3 - 3x, x^4 - 6x^2 + 1\}$

$B_v' = \{1, x, x^2 - 1, x^3 - 3x\}$ "Hermite polynomials"

$$[f]_{B_u', B_v} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -12 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

$$[f]_{B_u', B_v'} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

since $f(x^3 - 3x) = 3x^2 - 3 = 3(x^2 - 1)$

$f(x^4 - 6x^2 + 1) = 4x^3 - 12x = 4(x^3 - 3x)$

These alternative representations can also be computed using

$$[\text{identity}]_{B_u' B_u} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[\text{identity}]_{B_v, B_v'} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

since $x^3 = 3 \cdot x + 1 \cdot (x^3 - 3x)$

since $x^2 = 1 + (x^2 - 1)$

$$[f]_{B_u', B_v'} = [\text{identity}]_{B_v, B_v'} \cdot [f]_{B_u, B_v} \cdot [\text{identity}]_{B_u' B_u}$$

$$[f]_{B_U', B_V'} = [\text{identity}]_{B_V, B_V'} \, [f]_{B_U, B_V} \, [\text{identity}]_{B_U', B_U}$$

$$\mathcal{U}_{B_U'} \xrightarrow{\text{identity}} \mathcal{U}_{B_U} \xrightarrow{f} V_{B_V} \xrightarrow{\text{identity}} V_{B_V'}$$

Exercise: With the same $\mathcal{U}, V$ as above, consider the linear map $g: V \to \mathcal{U}$ given by

$$g(p) = (2+3x) \cdot p.$$

Give the matrices $[g]_{B_V B_U}$, $[g]_{B_V' B_U}$, $[g]_{B_V B_U'}$, $[g]_{B_V', B_U'}$.

(The most common mistake here is giving the <u>transposes</u> of the desired matrices.)

Example: Rowspace to columnspace

Consider the linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$ given, in the standard basis, by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow R(A) = \text{Span}\underbrace{\{(1,4,2), (2,0,-1)\}}_{B_V}$$
$$R(A^T) = \text{Span}\underbrace{\{(1,2,3), (1,0,1)\}}_{B_U}$$

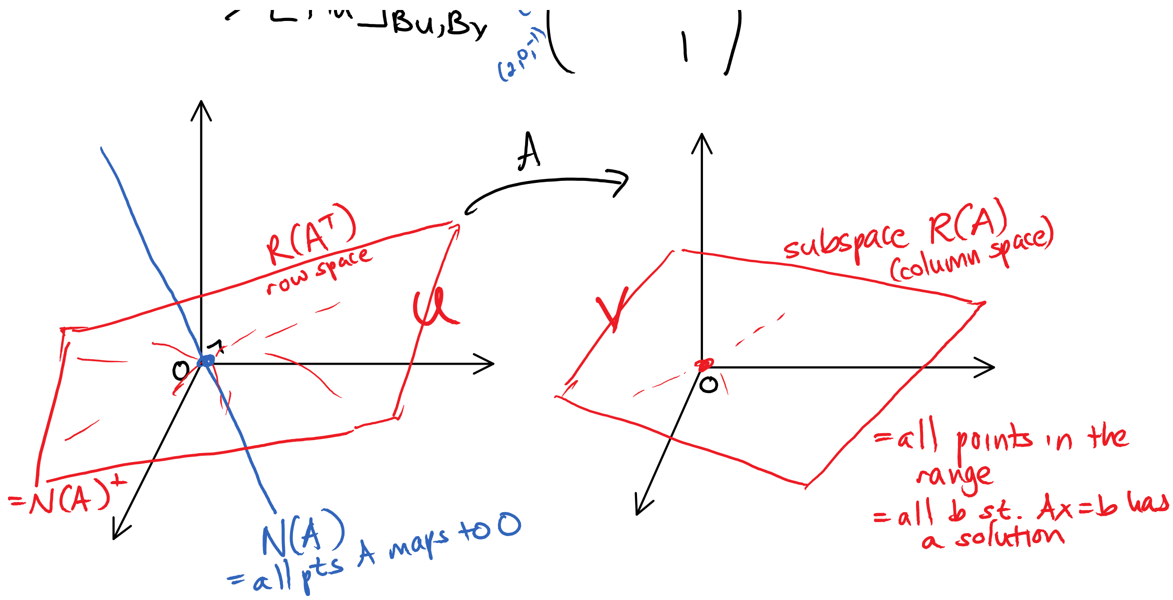
Let $\mathcal{U} = R(A^T)$, $V = R(A)$.

Then f restricted to vectors in $\mathcal{U}$ is still a linear transformation, which can be considered as mapping to V.

Q: What is $[f|_{\mathcal{U}}]_{B_U, B_V}$?

A: $f(1,2,3) = (14,16,3) = 4 \cdot (1,4,2) + 5 \cdot (2,0,-1)$
$f(1,0,1) = (4,8,3) = 2 \cdot (1,4,2) + (2,0,-1)$

$$\Rightarrow [f|_{\mathcal{U}}]_{B_U, B_V} = \begin{array}{c} \\ (1,4,2) \\ (2,0,-1) \end{array} \begin{array}{c} \begin{array}{cc} (1,2,3) & (1,0,1) \end{array} \\ \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix} \end{array}$$



Important concepts:

1. Span
2. Linear independence
3. Basis — orthogonal basis, change of basis
4. Dimension

① $\text{Span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\})$

$$= \left\{ \begin{array}{c} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r \end{array} \right\}$$

$$= R\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix} \right)$$ range/column space

② $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent if no vector $\vec{v}_j$ lies in the span of the others.

- equivalently, if the only solution to

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r = \vec{0}$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

- equivalently, if

$$\underbrace{N}_{\text{nullspace}}\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix}\right) = \{\vec{0}\}$$

③ A basis for a vector space V is a set of vectors that

- **spans** V, and
- is **linearly independent**

Equivalently, it is a minimal set of vectors that spans V.

④ Dimension(a vector space)

$= \#$ of vectors in a basis