Lecture 9: Rank-nullity, Graph flows

Tuesday, September 22, 2015 9:30 AM

Admin: Homework 3 due Thursday
Reading: Ch. 5 of Meyer, Chs. 2-3 Strang

Important concepts:

1. Span
2. Linear independence
3. Basis — orthogonal basis, change of basis
4. Dimension

① Span$(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\})$

$$= \left\{ \begin{array}{c} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r \end{array} \right\}$$

$$= R\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix}\right)$$ range/column space

② $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent if no vector $\vec{v}_j$ lies in the span of the others.

- equivalently, if the only solution to
$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r = \vec{0}$$
is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

- equivalently, if
$$\underbrace{N\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix}\right)}_{\text{nullspace}} = \{\vec{0}\}$$

③ A basis for a vector space V is a set of vectors that
- spans V, and
- is linearly independent

Equivalently, it is a minimal set of vectors that spans V.

④ Dimension (a vector space)
$= \#$ of vectors in a basis

Recall: A matrix corresponds to a linear transformation $\mathcal{U} \to V$, together with bases for $\mathcal{U}$ and V.

Matrix + bases $\longrightarrow$ Linear transformation

Linear transformation + bases $\longrightarrow$ Matrix

Today: Changing basis (continued)
Rank-nullity theorem

Example: Mapping rowspace to columnspace

Consider the linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$ given, in the standard basis, by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow R(A) = \text{Span}\overbrace{\{(1,4,2),(2,0,-1)\}}^{B_V}$$
$$R(A^T) = \text{Span}\overbrace{\{(1,2,3),(1,0,1)\}}^{B_{\mathcal{U}}}$$

Let $\mathcal{U} = R(A^T)$, $V = R(A)$.

Then f restricted to vectors in $\mathcal{U}$ is still a linear transformation, which can be considered as mapping to V.

Q: What is $[f|_{\mathcal{U}}]_{B_{\mathcal{U}},B_V}$?

A:
$$f(1,2,3) = (14,16,3) = 4\cdot(1,4,2) + 5\cdot(2,0,-1)$$
$$f(1,0,1) = (4,8,3) = 2\cdot(1,4,2) + (2,0,-1)$$
$$\Rightarrow [f|_{\mathcal{U}}]_{B_{\mathcal{U}},B_V} = \begin{array}{c|cc} & (1,2,3) & (1,0,1) \\ \hline (1,4,2) & 4 & 2 \\ (2,0,-1) & 5 & 1 \end{array}$$

CHANGES OF BASIS (continued)

Polynomial differentiation $\quad f(p) = \frac{d}{dx}p$

$$f: \left\{\begin{matrix}\text{polynomials in } x \\ \text{of degree} \leq 4\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{polynomials} \\ \text{of degree} \leq 3\end{matrix}\right\}$$

$\mathcal{U}$ (5-dim) $\qquad$ V (4-dim.)

$B_{\mathcal{U}} = \{1, x, x^2, x^3, x^4\}$ $\qquad$ $B_V = \{1, x, x^2, x^3\}$

$$[f]_{B_{\mathcal{U}},B_V} = \begin{array}{c|ccccc} & 1 & x & x^2 & x^3 & x^4 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 2 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 3 & 0 \\ x^3 & 0 & 0 & 0 & 0 & 4 \end{array}$$

since $f(1)=0$, $f(x)=1$, $f(x^2)=2x$, $f(x^3)=3x^2$, $f(x^4)=4x^3$

Different bases will give different matrices.

$B'_{\mathcal{U}} = \{1,\ x,\ x^2-1,\ x^3-3x,\ x^4-6x^2+1\}$

$B'_V = \{1,\ x,\ x^2-1,\ x^3-3x\}$ "Hermite polynomials"

Hermite polynomials

From Wikipedia, the free encyclopedia

In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence.

The polynomials arise in:

- probability, such as the Edgeworth series;

The first eleven probabilists' Hermite polynomials are:

$He_0(x) = 1$
$He_1(x) = x$
$He_2(x) = x^2 - 1$
$He_3(x) = x^3 - 3x$
$He_4(x) = x^4 - 6x^2 + 3$

In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence.

The polynomials arise in:

- probability, such as the Edgeworth series;
- in combinatorics, as an example of an Appell sequence, obeying the umbral calculus;
- in numerical analysis as Gaussian quadrature;
- in finite element methods as shape functions for beams;
- in physics, where they give rise to the eigenstates of the quantum harmonic oscillator;
- in systems theory in connection with nonlinear operations on Gaussian noise.

$$\begin{aligned}
He_2(x) &= x^2 - 1\\
He_3(x) &= x^3 - 3x\\
He_4(x) &= x^4 - 6x^2 + 3\\
He_5(x) &= x^5 - 10x^3 + 15x\\
He_6(x) &= x^6 - 15x^4 + 45x^2 - 15\\
He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x\\
He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105\\
He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x\\
He_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945
\end{aligned}$$

$$[f]_{B_u', B_v} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -12 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

$$[f]_{B_u', B_v'} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

since $f(x^3-3x) = 3x^2-3 = 3(x^2-1)$

$f(x^4-6x^2+1) = 4x^3-12x = 4(x^3-3x)$

These alternative representations can also be computed using

$$[\text{identity}]_{B_u' B_u} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[\text{identity}]_{B_v, B_v'} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

since $x^3 = 3 \cdot x + 1 \cdot (x^3-3x)$

since $x^2 = 1 + (x^2-1)$

$$[f]_{B_u', B_v'} = [\text{identity}]_{B_v, B_v'} \cdot [f]_{B_u, B_v} \cdot [\text{identity}]_{B_u' B_u}$$

$$\mathcal{U}_{B_u'} \xrightarrow{\text{identity}} \mathcal{U}_{B_u} \xrightarrow{f} \mathcal{V}_{B_v} \xrightarrow{\text{identity}} \mathcal{V}_{B_v'}$$

Example: Differentiation can also be considered as a map

{ polynomials in x } { polynomials in x }

$$\left\{\begin{array}{c}\text{polynomials in } x\\ \text{of degree} \le 4\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{polynomials in } x\\ \text{of degree} \le 4\end{array}\right\}$$

$$\qquad\quad \overset{\shortparallel}{\mathcal{U}} \qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathcal{U}}$$

basis $B = \{1, x, x^2, x^3, x^4\}$

$$[f]_B = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

since, eg.,

$$f(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0\cdot x^3 + 0\cdot x^4$$

basis $H = \{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$

basis change matrices:

$$[\text{identity}]_{H\to B} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[\text{identity}]_{B\to H} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

eg., $$x^4 = 1\cdot(x^4-6x^2+3) + 6\cdot(x^2-1) + 3\cdot 1$$

Observe: $[\text{identity}]_{B\to H} = \left([\text{identity}]_{H\to B}\right)^{-1}$

$$\texttt{idHtoB} = \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \ \texttt{idBtoH} = \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

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idHtoB.idBtoH // MatrixForm
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itrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{[f]_H = [id]_{B\to H}\,[f]_B\,[id]_{H\to B}} = [id]_{H\to B}^{-1}\,[f]_B\,[id]_{H\to B}$$

$$\texttt{fB} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

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idBtoH.fB.idHtoB // MatrixForm
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trixForm=

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for example,

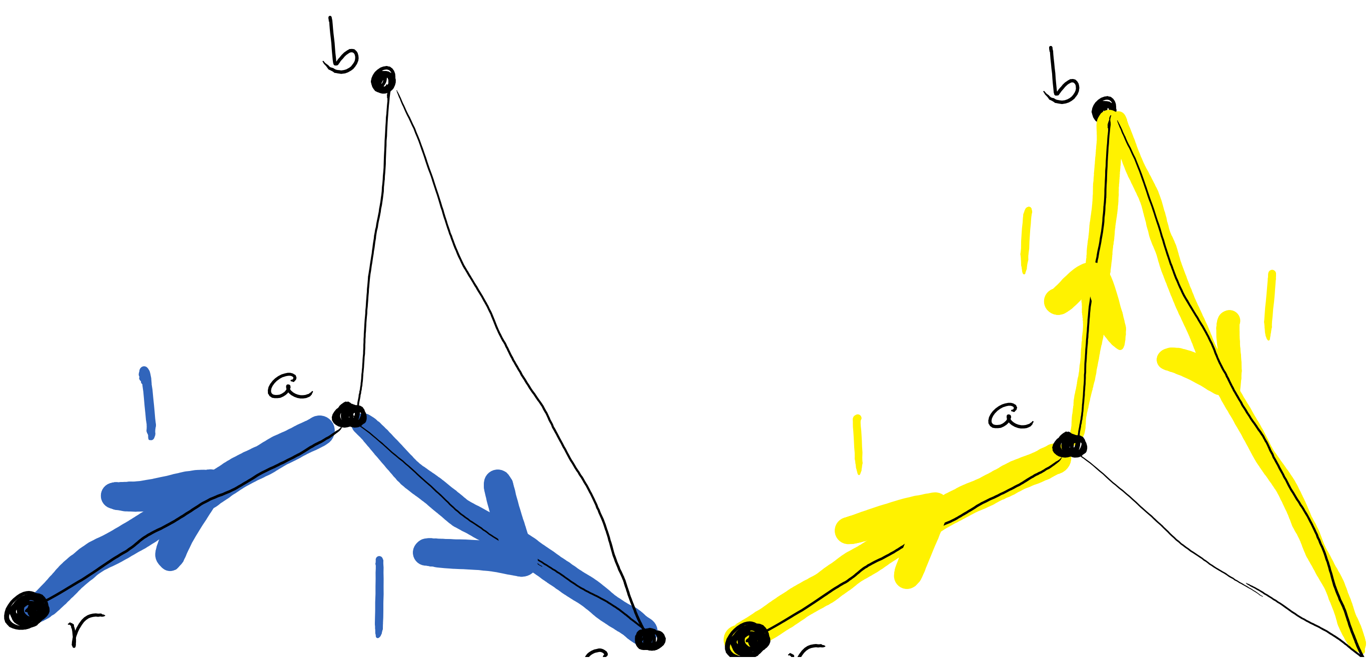
$$\frac{d}{dx}(x^4 - 6x^2 + 3) = 4x^3 - 12x = 4\cdot(x^3 - 3x)$$

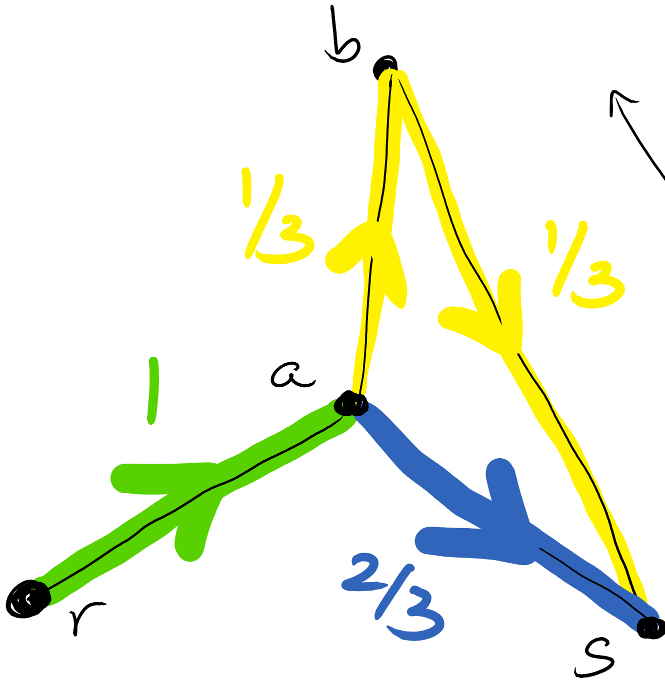
Note: Any nonsingular matrix A can be thought of as a basis change. A^{-1} changes back.

$$A \begin{pmatrix}\text{linear} \\ \text{trans.} \\ \text{in std. basis}\end{pmatrix} A^{-1} = \begin{pmatrix}\text{same linear} \\ \text{trans. in the} \\ \text{new basis}\end{pmatrix}$$

Rank-Nullity Theorem
Graph Flows

Example: Flows in a Graph





4 edges $\Rightarrow$ flow is a vector in $\mathbb{R}^4$

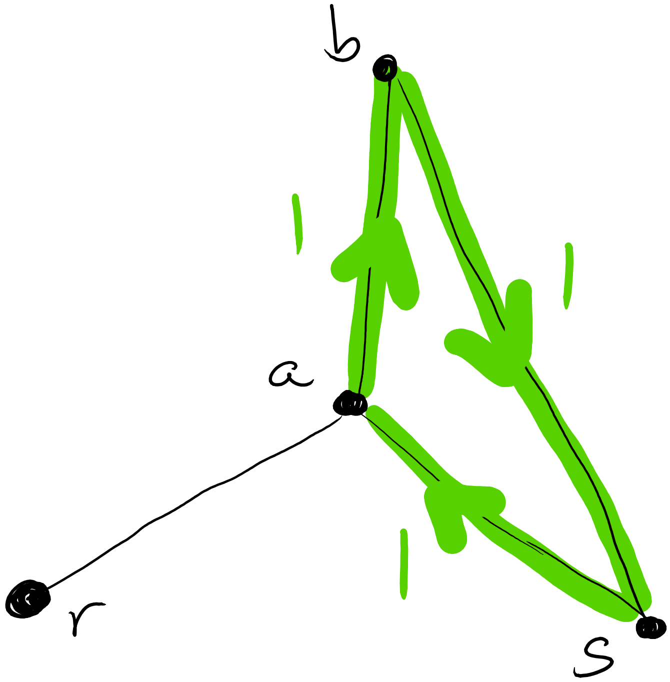
$$\begin{array}{cccc} (r,a) & (a,s) & (b,a) & (b,s) \end{array}$$

$$(1,\ 1,\ 0,\ 0)$$

$$(1,\ 0,\ -1,\ 1)$$

$$(1,\ 2/3,\ -1/3,\ 1/3)$$

unit r-s flows form a 1D affine subspace of $\mathbb{R}^4$
source sink



generated by cycle

$$(0,\ -1,\ -1,\ 1)$$

More generally:

graph $G=(V,E)$

n vertices m edges

Define the incidence matrix:

$$E_G = \begin{array}{c} \\ r \\ a \\ b \\ s \end{array} \begin{array}{c} \begin{array}{cccc} (r,a) & (a,s) & (b,a) & (b,s) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \end{array}$$

n rows

m columns

- A flow vector f is **balanced** everywhere if for every vertex, the total entering that vertex is 0.
  Equivalently, $f \in N(E_G)$.
- Vertex r is connected to $s \iff e_r - e_s \in R(E_G)$.

__Problems__: For a general graph G,

1. What is $\dim N(E_G)$?
   ... a basis for $N(E_G)$?
2. What is $\dim R(E_G)$?
   ... a basis?

Why care?

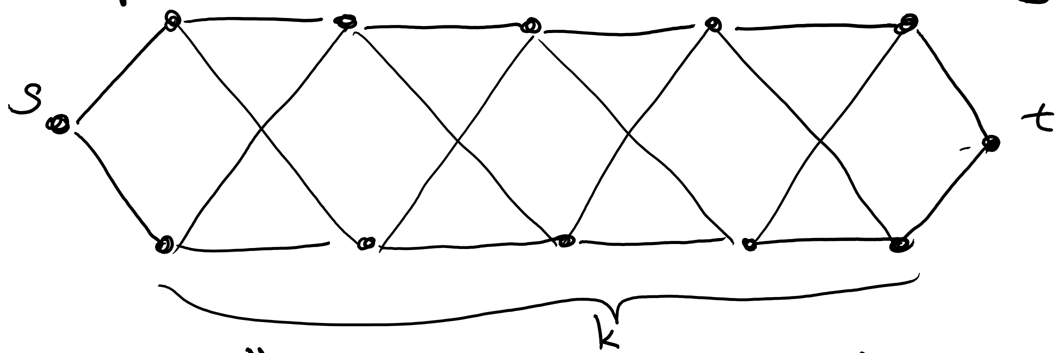
- Closely related to connectivity problems on G
- Problems involving flows are everywhere,
  e.g., electricity,
  routing (oil, internet, ...)

But there are infinitely many flows!
Even if we want to optimize a nonlinear cost

function, a basis for the space lets us use calculus, reduces complexity.

eg. energy dissipated by electrical flows $= \sum_{\text{edges } e} (\text{resistance of } e) \cdot (\text{flow through } e)^2$

Example: Graph flows and connectivity



exponentially many left-right paths from s to t
infinitely many flows
$\Rightarrow$ we need a simple basis

To solve these problems, we need one more tool:

RANK-NULLITY THEOREM

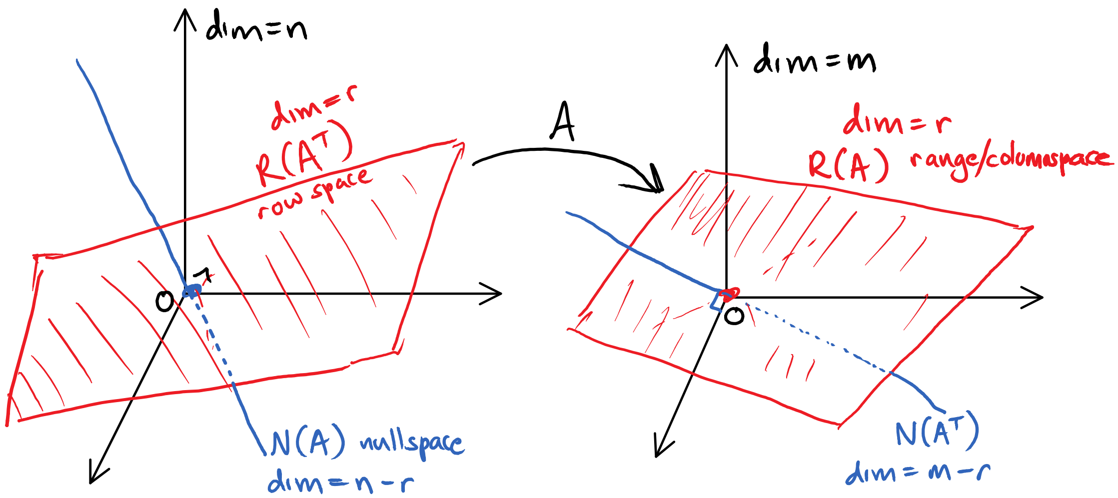
Definition: The rank of a matrix A is

$$\text{rank}(A) = \dim R(A),$$

the dimension of the range.

Rank-Nullity Theorem:
Let A be an $m \times n$ matrix. Then
- $\dim R(A^T) = \dim R(A)$
- $\dim N(A) = n - \dim R(A)$
- $\dim N(A^T) = m - \dim R(A)$.

Intuition:



$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\begin{array}{l}\dim N(A) + \dim R(A^T) \\ \quad = \text{total dimension } n\end{array}} \qquad \boxed{\begin{array}{l}\dim R(A) + \dim N(A^T) \\ \quad = \text{total dimension } m\end{array}}$$

Corollary: A is invertible $\Leftrightarrow$ $m = n = \text{rank}(A)$.
"full rank"

Example:
Problem: Give a basis for $\underbrace{\{\vec{x} \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 0\}}_{\text{"}V\text{"}}$.

Answer:

$$V = N(\underbrace{(1 \ 1 \ 1 \ \cdots \ 1)}_{\text{"}A\text{"}})$$

$$\text{rank}(A) = 1$$

$$\Rightarrow \dim N(A) = n - \text{rank}(A) \qquad \text{[Rank-Nullity]}$$

$$= n - 1$$

$\Rightarrow$ Any linearly independent set of $n-1$ elements of V is a basis for V.

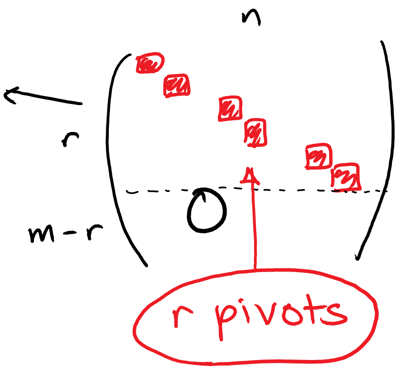
For example,
$\{\vec{e}_1 - \vec{e}_n, \vec{e}_2 - \vec{e}_n, \vec{e}_3 - \vec{e}_n, \ldots, \vec{e}_{n-1} - \vec{e}_n\}$ is a basis. ✓

Proof of Rank-Nullity Theorem:

As usual, use Gaussian elimination

$A = G\ U$

invertible row operations



1. $\dim R(U^T) = \dim \mathrm{Span}(\text{rows}) = r$:
   because the first r rows are linearly independent, the other rows are 0

2. $\dim N(U^T) = m - r$:
   left nullspace
   - it is $\geq m-r$ since any combination of the last $m-r$ rows gives 0
   - but no nontrivial combination of the first r rows gives 0 (linear independence).

3. $\dim R(U) = \dim \mathrm{Span}(\text{columns}) = r$
   - it is $\geq r$ since the pivot columns ("basic columns") are linearly independent
   - it is $\leq r$ since any column is in the span of the pivot columns.

4. $\dim N(U) = n - r$
   nullspace
   since there are $n-r$ free variables (the non-pivot columns) when we solve $Ux = 0$.

$A = G\ U$

Note: G is invertible, so it maps linearly independent sets of vectors to lin. indep. sets.

$\Rightarrow \dim R(A) = \dim R(U) = r$

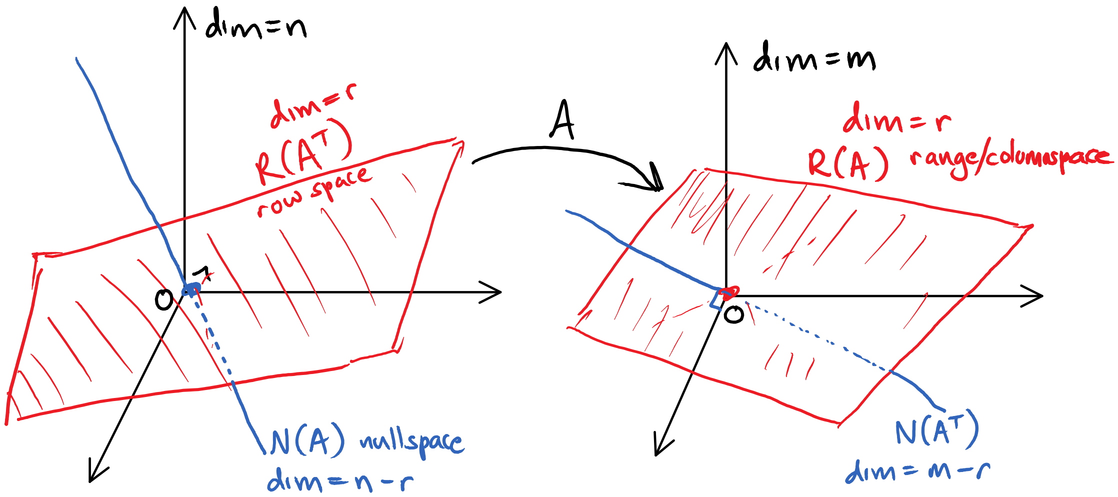
$G(R(U))$

$N(A) = N(U)$, $R(A^T) = R(U^T)$

$N(A^T) = G^{-1}(N(U^T))$ have same dimension.

$\square$

Rank-Nullity Thm.



$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\begin{aligned}&\dim N(A) + \dim R(A^T)\\ &= \text{total dimension } n\end{aligned}}$$

$$\boxed{\begin{aligned}&\dim R(A) + \dim N(A^T)\\ &= \text{total dimension } m\end{aligned}}$$

EXAMPLE: FLOWS IN A GRAPH (cont.)

Theorem: For any graph G — n vertices, m edges

(*) $$\operatorname{rank}({}_{n}^{m}E_G) = n - \#\text{connected components of } G.$$

Corollary: G is connected $\iff \operatorname{rank}(E_G) = n-1$.

Corollary: $\dim N(E_G) = m - n + \#\text{components}$.

balanced flows

Proof: We will show

$\dim N(E_G^T) = \#$ connected components

The Rank-Nullity Theorem then implies $(*)$.

$$E_G^T = \begin{array}{c} \\ (u,v) \\ (u,w) \\ (v,w) \\ \vdots \\ m \text{ edges} \end{array} \begin{array}{c} \begin{array}{ccccc} u & v & w & & n \text{ vertices} \end{array} \\ \left(\begin{array}{ccccc} 1 & -1 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

For $x \in \mathbb{R}^n$ and a vertex v, let $x_v =$ component of x in the v dimension

$y \in \mathbb{R}^m$, " edge e, $y_e =$ component in e dim.

$x \in N(E_G^T)$

$\iff E_G^T x = 0$

$\iff$ for every edge $e = (u,v)$,

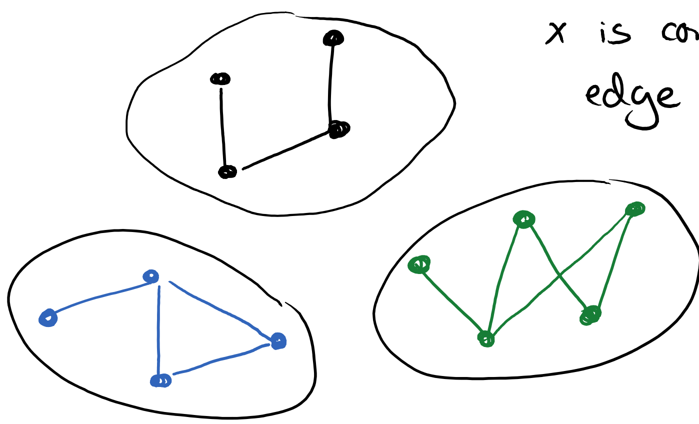
$(E_G^T x)_e = 0 = x_u - x_v$

$\iff$ for every edge (u,v),

$x_u = x_v$

x is constant across edges!

Here's G, with 3 connected components:



x is constant across each edge $\iff$ constant across each component

e.g., $(1,1,1,1,2,2,2,2,-1,-1,-1,-1,-1) \in N(E_G^T)$

$\Rightarrow \dim N(E_G^T) = \#$connected components. ✓ $\square$

Basis for $N(E_G^T)$: $(1,1,1,1,0,0,0,0,0,0,0,0,0)$

$(0,0,0,0,1,1,1,1,0,0,0,0,0)$
$(0,0,0,0,0,0,0,0,1,1,1,1,1)$

Bases for $N(E_G)$ and $R(E_G)$?

If G splits as

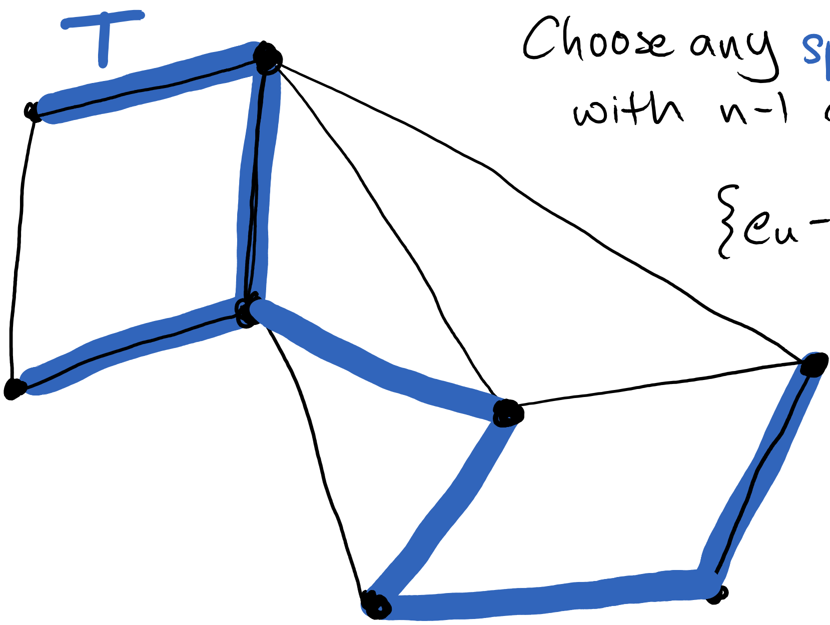
G_1 G_2 G_3 $\Rightarrow E_G = \begin{pmatrix} E_{G_1} & 0 & 0 \\ 0 & E_{G_2} & 0 \\ 0 & 0 & E_{G_3} \end{pmatrix}$

up to reordering rows & columns

$\Rightarrow$ We can combine the answers $N(E_{G_i}), R(E_{G_i})$

Assume G is connected.

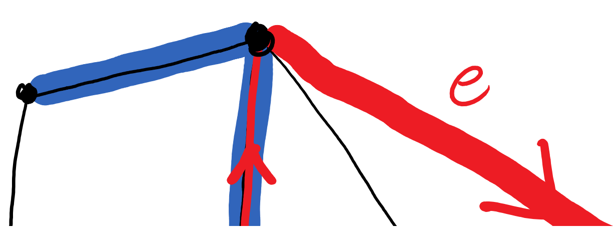
$\dim N(E_G) = m - n + 1$
$\dim R(E_G) = n - 1$

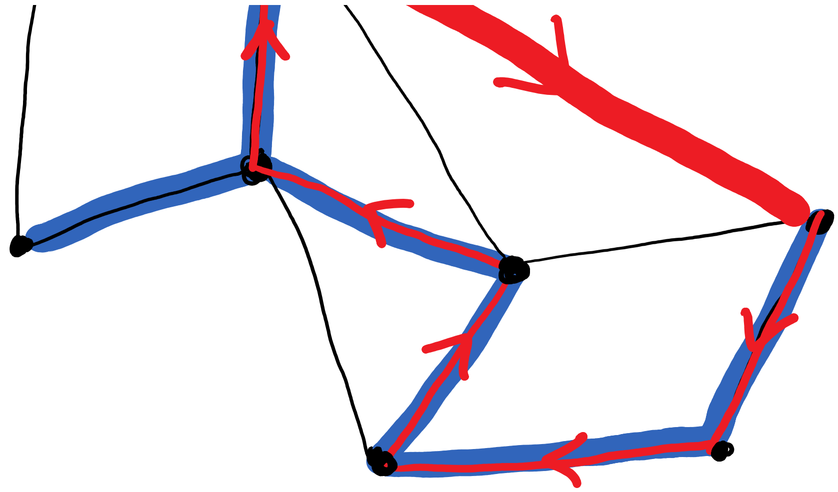


Choose any spanning tree T, with $n-1$ edges

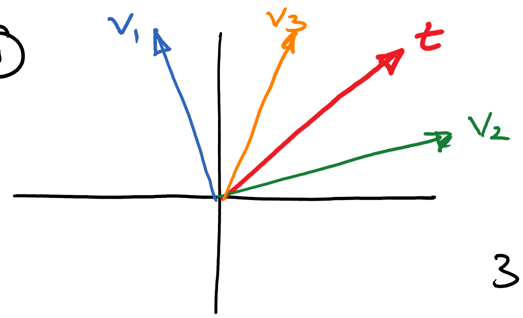
$\{e_u - e_v \mid (u,v) \in T\}$ is linearly indep., hence a basis for $R(E_G)$

For any edge $e \notin T$, the cycle through T and e is a balanced flow, and these cycles are lin. ind.
$m-n+1$ cycles $\longrightarrow$ a basis for $N(E_G)$.





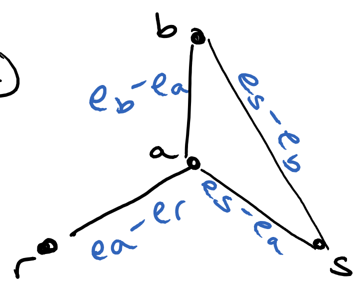
EXAMPLE: "SPAN PROGRAMS"

① 

target input vectors
$t, v_1, v_2, v_3 \in \mathbb{R}^2$

Observe: The "target" t lies in the span of any 2 of the 3 other vectors (the majority).

The pattern of which subsets of input vectors span the target $\longrightarrow$ a function's truth table (majority). The vectors form a "span program."

② 

$$\mathbb{R}^4: \quad \begin{aligned} e_r &= (1,0,0,0) \\ e_s &= (0,1,0,0) \\ e_a &= (0,0,1,0) \\ e_b &= (0,0,0,1) \end{aligned}$$

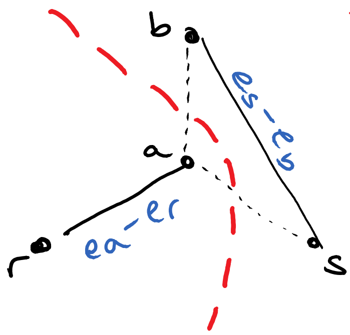
target $e_s - e_r = (-1,1,0,0)$
$$= (e_a - e_r) + (e_b - e_a) + (e_s - e_b)$$

path 

Any path from r to s gives a way of spanning the target.

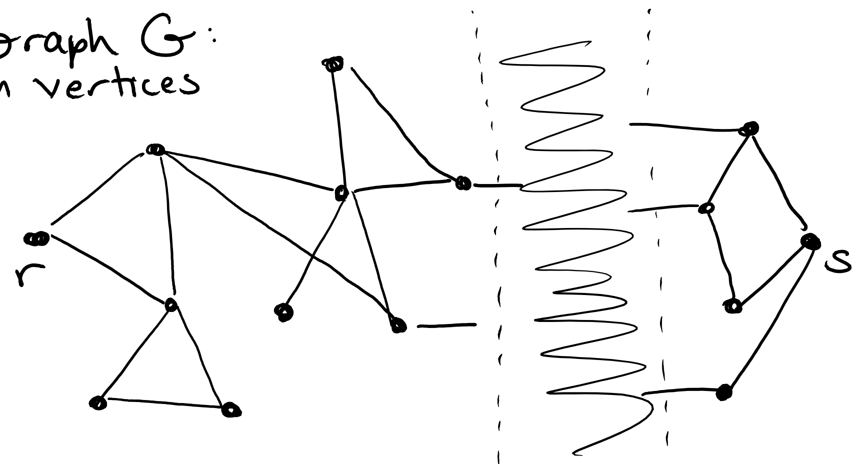
But if we cut the graph,

target $e_s - e_r \notin \text{Span}\begin{pmatrix} e_a - e_r, \\ e_s - e_b \end{pmatrix}$



More generally:

Graph G:
n vertices



Question: Is there a path from r to s?

Vector space $\mathbb{R}^n$ vertex $v \longleftrightarrow e_v = (0 \cdots \underset{\substack{\uparrow \\ \text{coordinate } v}}{1} \cdots 0)$

target $t = e_s - e_r$ edge $(u,v) \longleftrightarrow e_u - e_v$

target $\in \text{Span}\begin{pmatrix}\text{edge} \\ \text{vectors}\end{pmatrix} \iff$ r is connected to s in G

"FACT": Span programs $\longleftrightarrow$ Quantum algorithms

(weird, but algebraically simple model) (physically natural, but complicated model)

The above span program gives the fastest, most space-efficient algorithm known for determining if r is connected to s.