Homework 4

Thursday, September 24, 2015 9:30 AM

Due in class Thursday 10/1.

① For a general vector space, let U and V be two subspaces with respective bases $B_U = \{\vec{u}_1, \dots, \vec{u}_m\}$ and $B_V = \{\vec{v}_1, \dots, \vec{v}_n\}$.
Prove that
$U \cap V = \{\vec{0}\}$ if and only if $\{\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_n\}$ is a linearly independent set.

② Find two different unit vectors (i.e., length-one vectors) that are orthogonal to $\vec{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbb{R}^2$.

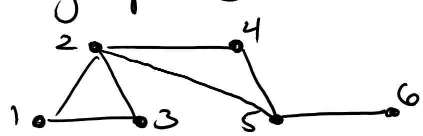
③ Let

$$A = \begin{pmatrix} 2 & 2 & 5 & 0 & 1 \\ 3 & 4 & 8 & 1 & 2 \\ 1 & 6 & 5 & 5 & 3 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 3 \\ -8 \end{pmatrix}.$$

Verify that $v \in N(A)$, and then extend $\{v\}$ to a basis for $N(A)$. (That is, find a basis for $N(A)$ that includes v as one of its elements.)

④ Recall from Lecture 9 the edge incidence matrix E_G for a graph G.
(If G has n vertices, m edges, then E_G is an $n \times m$ matrix.)

ⓐ For the graph G



what is E_G? (Orient the edges whichever way you like.)

ⓑ For the same graph, what is $E_G E_G^T$?
What is the rank of $E_G E_G^T$? Feel free to use Matlab/Mathematica.
What is the nullspace of $E_G E_G^T$?
(Hint: Recall Homework 2 #9.)

ⓒ For a general graph G, compute the $n \times n$ matrix $E_G E_G^T$.
There should be 3 cases:

1) If vertex i is not adjacent to vertex j, argue that $(E_G E_G^T)_{i,j} = 0$.

2) What is $(E_G E_G^T)_{i,i}$?

3) What is $(E_G E_G^T)_{i,j}$ when i is adjacent to j?

If G is connected, what is $R(E_G E_G^T)$?

5) What is the projection of $b = (2,1,2)$ onto the plane spanned by $(0,1,0)$ and $(0,1,1)$?

6) Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that

- First rotates the xz-plane by $\pi/3$ radians counterclockwise about the y-axis,
- Then reflects everything about the yz-plane (i.e., switching the sign of the x coordinate),
- Then rotates the xy-plane by $\pi/6$ radians counterclockwise about the z-axis.

What is the 3×3 matrix representing f?
What is the determinant of this matrix?

7) a) Prove the "parallelogram law":

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2).$$

b) Show the implications:

$$\begin{tikzcd}
 & \vec{x} \text{ is orthogonal to } \vec{y} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
\vec{x}+\vec{y} \text{ and } \vec{x}-\vec{y} \text{ have the same length} \arrow[rr, Leftrightarrow] & & \|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2
\end{tikzcd}$$

8) Consider the 2×2 complex matrix

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Rewrite this matrix in the basis $\{(1,i),(1,-i)\}$.

Simplify your answer as much as possible.

(That is, give a 2×2 matrix that corresponds to the same linear transformation as A, but in the basis $\{(1,i),(1,-i)\}$ instead of the standard basis $\{e_1, e_2\}$.)