Lecture 12: Orthogonal bases

Thursday, October 1, 2015 9:30 AM

Admin: Midterm October 8.

Recall: The orthogonal projection of $\vec{u}$ onto the line Span($\vec{v}$) is given by $\boxed{\frac{v v^T}{\|v\|^2}\vec{u} = \frac{(v \cdot u)\, v}{\|v\|^2}}$

Example: In $\mathbb{R}^3$, $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

$$e_1 e_1^T = \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

projects onto the x-axis.

Exercise: What is the projection of

$$\vec{u} = (2, -2, 3)$$

onto the line

$$L = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x + 2y + 3z = 0 \\ x - y + 2z = 0 \end{array} \right\} \quad ?$$

Answer:

① First find a basis for L.

L is a line because it is the intersection of two 2D planes in $\mathbb{R}^3$; it is the nullspace of the matrix

$$A = \begin{pmatrix}1 & 2 & 3\\1 & -1 & 2\end{pmatrix}.$$

Use Gaussian elimination to find $N(A)$:

$$A \longrightarrow \begin{pmatrix}1 & 2 & 3\\0 & -3 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 7/3\\0 & 3 & 1\end{pmatrix}$$

$\Rightarrow x = -7/3 \cdot z, \quad y = -\frac{1}{3} z$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}7\\1\\-3\end{pmatrix} z$ is the general solution to

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\Rightarrow N(A) = \text{Span}\{(7, 1, -3)\}$$

(Check: Yes, $A\begin{pmatrix}7\\1\\-3\end{pmatrix} = 0$.)

② Normalize it: $\|(7,1,-3)\|^2 = 59$

$\Rightarrow \vec{v} = \frac{1}{\sqrt{59}}(7,1,-3)$ is a unit vector

③ Project

$$P_L\vec{u} = (\vec{v}\,\vec{v}^T)\vec{u}$$
$$= \vec{v}(\vec{v}^T\vec{u})$$
$$= (\vec{v}\cdot\vec{u})\vec{v}$$
$$= \frac{1}{\sqrt{59}}(7\cdot 2 + 1\cdot(-2) - 3\cdot 3)\frac{1}{\sqrt{59}}(7,1,-3)$$

$$\boxed{P_L\vec{u} = \frac{3}{59}(7,1,-3)}$$

④ Check the answer:

$\frac{3}{59}(7,1,-3) \in L$: ✓ since it is a multiple of $\vec{v}$, and yes $x+2y+3z=0$, $x-y+2z=0$.

$(\vec{u} - P_L\vec{u}) \perp L$:

$$(2,-2,3) - \frac{3}{59}(7,1,-3) = \frac{1}{59}(97,-121,186)$$

$$(97,-121,86)\cdot(7,1,-3) = 0 \ ✓$$

Exercise: What is the projection of

$$\vec{u} = (4, 0, 3)$$

onto the line

$$L' = \left\{(x,y,z) \;\middle|\; \begin{array}{l} x+2y+3z = 6 \\ x-y+2z = 0 \end{array}\right\} ?$$

Answer:

① Find a constructive formulation for L':

$L' = $ (any particular solution) $+ N(A)$

to $A\begin{pmatrix}y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

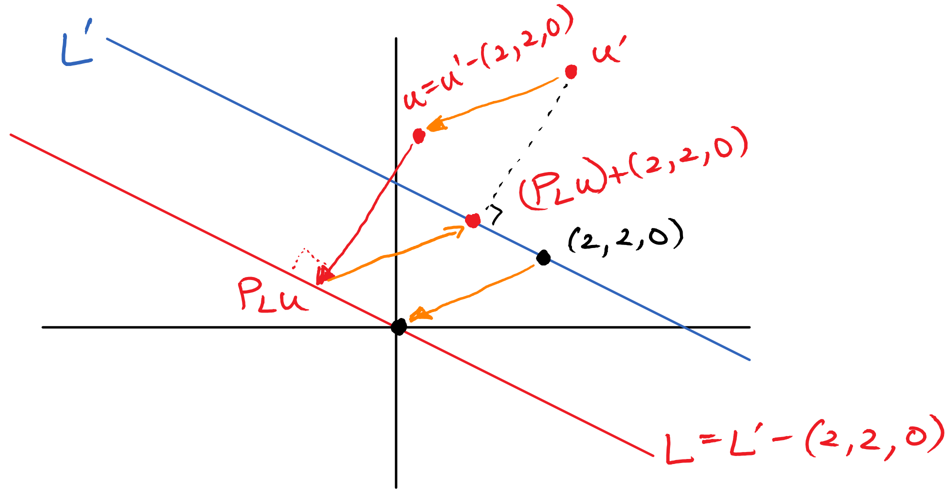
$= (2,2,0) + \mathrm{Span}((7,1,-3))$.

General principle: To work with an affine subspace,

A. translate everything so it goes through 0

B. work there

C. translate back!



Since $L' - (2,2,0) = L$ (from above)

and $u' - (2,2,0) = u$,

the projection of u' onto L' is

$$\boxed{(2,2,0) + \tfrac{3}{59}(7,1,-3) = \tfrac{1}{59}(139,121,-9)}.$$

Pairwise Orthogonal Sets of Vectors Are Nice!!

$\{\vec{v}_1, \dots, \vec{v}_n\}$

with $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$

① Simplifies deciding linear independence

Normally, to check if $S = \{v_1, \dots, v_n\}$ is lin. indep., compute $N\left(\left(v_1 \mid \cdots \mid v_n\right)\right) \stackrel{?}{=} \{0\}$.

Lemma:

$S = \{v_1, v_2, \ldots, v_n\}$ non-zero, pairwise orthogonal $v_i \cdot v_j = 0$ for $i \neq j$ $\Rightarrow$ S is linearly independent

Proof: Assume $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = 0$.

$\Rightarrow \alpha_1(\vec{v}_i \cdot v_1) + \alpha_2(\vec{v}_i \cdot \vec{v}_2) + \cdots + \alpha_n(\vec{v}_i \cdot \vec{v}_n) = 0$

$\| \quad \alpha_i \vec{v}_i \cdot \vec{v}_i = \alpha_i \|v_i\|^2$

$\Rightarrow \alpha_i = 0$ (since $v_i \neq 0$) for all i $\square$

② Pairwise $\perp$ity simplifies deciding if S is a basis

Corollary: In an n-dimensional vector space, any set of n pairwise orthogonal vectors forms a basis.

(because any n linearly indep. vectors form a basis)

Definition: Orthogonal basis = basis of pairwise orthogonal vectors

Orthonormal basis = basis of pairwise orthogonal, length-one vectors

Example: The standard basis for $\mathbb{R}^n$ is orthonormal.

Dividing by the lengths, orthogonal $\rightarrow$ orthonormal basis.

③ Simplifies computing inner products & lengths

Let $\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_m \vec{v}_m$

<u>Q</u>: What is $\|u\|$?

<u>A</u>: $$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$$
$$= \left(\sum_i \alpha_i \vec{v}_i\right) \cdot \left(\sum_j \alpha_j \vec{v}_j\right)$$
$$= \sum_{i,j=1}^{n} \alpha_i^* \alpha_j (\vec{v}_i \cdot \vec{v}_j)$$

n^2 terms to sum up!

But <u>if</u> $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is <u>orthogonal</u>,
$$\|\vec{u}\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \cdot \|\vec{v}_i\|^2$$
since $\vec{v}_i \cdot \vec{v}_j$ terms are otherwise 0,

And <u>if</u> $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is <u>orthonormal</u>,
$$\boxed{\|u\|^2 = \sum_{i=1}^{n} |\alpha_i|^2}$$
the exact same expression as works in the standard basis

<u>MORAL</u>: Orthonormal bases behave just like the standard basis.

e.g., for $\vec{u} = \sum_j \alpha_j \vec{v}_j$, $\vec{v} = \sum_j \beta_j \vec{v}_j$,
$$\vec{u} \cdot \vec{v} = \sum_j \alpha_j^* \beta_j$$

④ Simplifies basis expansions

Let $B = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ be a basis for $V \subseteq \mathbb{R}^n$,
and $\vec{u} = (u_1, \ldots, u_n) \in V$.

What is the expansion of $\vec{u}$ in the basis B?

<u>In general</u>: Write
$$_n\left(\vec{u}\right) = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}_m \begin{pmatrix} | \\ x \\ | \end{pmatrix}$$

In general: Write

$$\begin{pmatrix} \vec{u} \end{pmatrix}_n = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m = {}_n\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix}$$

and solve these equations.

If B is orthonormal:

$$\vec{u} = x_1 \vec{v}_1 + \cdots + x_j \vec{v}_j + \cdots + x_m \vec{v}_m$$

$$\Downarrow$$

$$\vec{v}_j \cdot \vec{u} = \vec{v}_j \cdot \left(x_1 \vec{v}_1 + \cdots + x_j \vec{v}_j + \cdots + x_m \vec{v}_m \right)$$

$$= 0 + 0 + \cdots + x_j + \cdots + 0$$

$$= x_j$$

$$\Rightarrow \boxed{\vec{u} = \sum_{j=1}^{m} (\vec{v}_j \cdot \vec{u})\, \vec{v}_j}$$

- just compute these inner products
- needn't solve any equations!

Example: Haar wavelet basis for $\mathbb{R}^8$

v_1	(1, 1, 1, 1, 1, 1, 1, 1)
v_2	(1, 1, 1, 1, −1, −1, −1, −1)
v_3	(1, 1, −1, −1, 0, 0, 0, 0)
v_4	(1, −1, 0, 0, 0, 0, 0, 0)
v_5	(0, 0, 1, −1, 0, 0, 0, 0)
v_6	(0, 0, 0, 0, 1, 1, −1, −1)
v_7	(0, 0, 0, 0, 1, −1, 0, 0)
v_8	(0, 0, 0, 0, 0, 0, 1, −1)

Pairwise $\perp$

$\Downarrow$

Linearly independent

$\Downarrow$

Must be a basis!

Exercise: Expand $\overset{u}{(1, 1, 1, 0, -2, -2, -2, -2)}$ in the above basis.

Answer:

$v_1 \cdot u = -5$

$v_2 \cdot u = 11$

$v_3 \cdot u = 1$

$v_4 \cdot u = 0$

$v_5 \cdot u = 1$

$\vdots$

$$\Rightarrow \vec{u} = \sum_j \frac{(\vec{v}_j \cdot \vec{u})\vec{v}_j}{\|v_j\|^2}$$

$$= -\frac{5}{8}\vec{v}_1 + \frac{11}{8}v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_5 + \cdots$$

We could compute $v_6 \cdot u$, $v_7 \cdot u$, $v_8 \cdot u$ — and its easy — but in fact, observe

$$\|u\|^2 = 19$$

squared coeff. of $v_2/\|v_2\|$

and
$$\frac{|v_1\cdot u|^2}{\|v_1\|^2} + \frac{|v_2\cdot u|^2}{\|v_2\|^2} + \frac{|v_3\cdot u|^2}{\|v_3\|^2} + \frac{|v_5\cdot u|^2}{\|v_5\|^2}$$

$$= \frac{25}{8} + \frac{121}{8} + \frac{1}{4} + \frac{1}{2}$$

$$= 19$$

$\Rightarrow$ all other coefficients **must** be 0, since all of $\vec{u}$'s length is accounted for!

(This is a common trick, to save time.)

④ Simplifies matrix basis expansions

$$f : U \to V$$

$B_U = \{\vec{u}_1, \ldots, \vec{u}_n\}$

$B_V = \{\vec{v}_1, \ldots, \vec{v}_m\}$ orthonormal basis

$$\Rightarrow [f]_{B_U \to B_V} = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & \vdots & \\ \cdots & \vec{v}_i \cdot f(\vec{u}_j) & \\ & & \end{pmatrix} \end{array}$$

since $f(\vec{u}_j) = \sum_{i=1}^{m} (\vec{v}_i \cdot f(\vec{u}_j))\, \vec{v}_i$

⑤ Simplifies changing basis

Recall: $f : U \to V$ linear

U: bases B_U, B_U'

V: bases B_V, B_V'

$$[f] = [I]\,[f]\,[I]$$

$$[f]_{B'_U \to B'_V} = [I]_{B_V \to B'_V} \, [f]_{B_U \to B_V} \, [I]_{B'_U \to B_U}$$

$$\underset{B'_V}{V} \xleftarrow{I_V} \underset{B_V}{V} \xleftarrow{f} \underset{B_U}{U} \xleftarrow{I_U} \underset{B'_U}{U}$$

if $U = V$,

$$[f]_{B' \to B'} = \underbrace{[I]_{B \to B'}}_{= \left([I]_{B' \to B}\right)^{-1}} [f]_{B \to B} [I]_{B' \to B}$$

Example:

Consider the 2×2 complex matrix

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Write this matrix in the basis $\left\{ \frac{1}{\sqrt{2}}(1, i), \frac{1}{\sqrt{2}}(1, -i) \right\}$.

Answer: Initial basis $B = \{(1,0), (0,1)\}$.

New basis $C = \left\{ \frac{1}{\sqrt{2}}(1, i), \frac{1}{\sqrt{2}}(1, -i) \right\}$.

Observe: B' is orthonormal!

$$\left\| \tfrac{1}{\sqrt{2}}(1, \pm i) \right\|^2 = \tfrac{1}{2} \left\| (1, \pm i) \right\|^2$$

$$= \tfrac{1}{2}\left(1^2 + |\pm i|^2\right) = 1 \checkmark$$

$$\tfrac{1}{\sqrt{2}}(1, i) \cdot \tfrac{1}{\sqrt{2}}(1, -i) = \tfrac{1}{2}\left(1 + i^*(-i)\right)$$

$$= \tfrac{1}{2}\left(1 + i^2\right) = 0 \checkmark$$

$C \to B$ basis change:

$$\begin{array}{c c} & \begin{array}{cc} \tfrac{1}{\sqrt{2}}(1,i) & \tfrac{1}{\sqrt{2}}(1,-i) \end{array} \\ \begin{array}{c} (1,0) \\ (0,1) \end{array} & \begin{pmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \\ \tfrac{i}{\sqrt{2}} & \tfrac{-i}{\sqrt{2}} \end{pmatrix} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$B \to C$ basis change

$$\begin{array}{c c} & \begin{array}{cc} (1,0) & (0,1) \end{array} \\ \begin{array}{c} \tfrac{1}{\sqrt{2}}(1,i) \end{array} & \begin{pmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{-i}{\sqrt{2}} \end{pmatrix} \end{array} \quad \frac{1}{\ } \begin{pmatrix} 1 & -i \end{pmatrix}$$

B → C basis change

$$\begin{array}{c|cc} & (1,0) & (0,1) \\ \hline \frac{1}{\sqrt2}(1,i) & \frac{1}{\sqrt2} & \frac{-i}{\sqrt2} \\ \frac{1}{\sqrt2}(1,-i) & \frac{1}{\sqrt2} & \frac{i}{\sqrt2} \end{array} = \frac{1}{\sqrt2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

since, eg., $\frac{1}{\sqrt2}(1,i)\bullet(0,1) = \frac{-i}{\sqrt2}$

<u>Check</u>: $(1,0) = \frac{1}{\sqrt2}\left(\frac{1}{\sqrt2}(1,i)+\frac{1}{\sqrt2}(1,-i)\right)$ ✓

$(0,1) = \frac{1}{\sqrt2}\left(\frac{-i}{\sqrt2}(1,i)+\frac{i}{\sqrt2}(1,-i)\right)$ ✓

$$\Rightarrow [A]_{C\to C} = [B\to C]\cdot[A]_{B\to B}\cdot[C\to B]$$

(read this right to left)

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

In[1]:= $\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\cdot\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ // FullSimplify // MatrixForm

Out[1]//MatrixForm=

$$= \boxed{\begin{pmatrix} \text{Cos}[\theta] & i\,\text{Sin}[\theta] \\ i\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}}$$

(Mathematica)

<u>Observe</u>: 1. This was really easy!

2. $\frac{1}{\sqrt2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

— they're inverses

3. They differ only by transpose and complex conjugate.

In general,

<u>If B, C are each orthonormal</u>:

$B = \{\vec b_1, \ldots, \vec b_n\}$, $C = \{\vec c_1, \ldots, \vec c_n\}$.

$$[I]_{C\to B} = \begin{array}{c} \\ i \end{array}\begin{array}{c} \vec c_j \\ \begin{pmatrix} & \vdots & \\ \cdots & \boxed{\vec b_i\bullet\vec c_j} & \\ & \vdots & \end{pmatrix} \end{array} \qquad \text{since } \vec c_j = \sum_{i=1}^{n}(b_i\bullet c_j)\vec b_i$$

$$[I]_{C\to B} = \begin{array}{cc} & \vec{c}_j \\ i & \begin{pmatrix} & \vert & \\ \cdots & \boxed{\vec{b}_i\cdot\vec{c}_j} & \\ & \vert & \end{pmatrix} \end{array} \qquad \text{since } \vec{c}_j = \sum_{i=1}^{n} (b_i\cdot c_j)\,\vec{b}_i$$

$$[I]_{B\to C} = \begin{array}{cc} & \vec{b}_i \\ j & \begin{pmatrix} & \vert & \\ \cdots & \boxed{\vec{c}_j\cdot\vec{b}_i} & \\ & \vert & \end{pmatrix} \end{array} \qquad \text{since } \vec{b}_i = \sum_{j=1}^{n} (c_j\cdot b_i)\,\vec{c}_j$$

$$\Rightarrow \boxed{[I]_{B\to C} = \left([I]_{C\to B}\right)^{\dagger}}$$

"adjoint"
= transpose
+ complex conjugate
(same as transpose for real matrices)

⑦ Simplifies computing projections

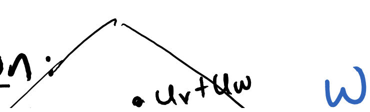
Problem: How can we project onto a higher-dimensional subspace?

Example: $P_{xy\text{-plane}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

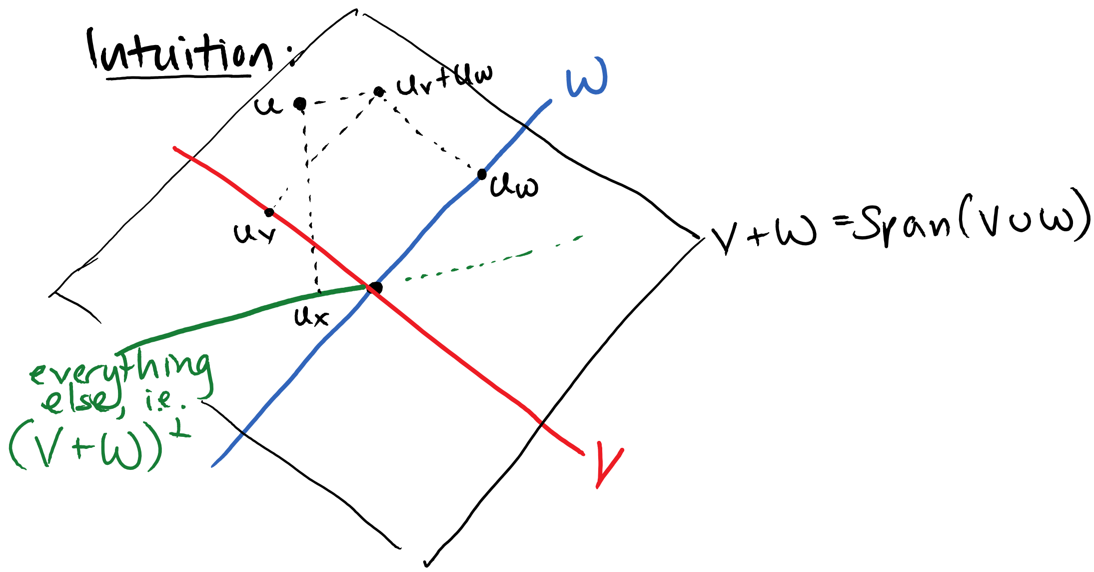
$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{P_{x\text{-axis}} = \begin{pmatrix}1\\0\\0\end{pmatrix}(1\ 0\ 0)} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{P_{y\text{-axis}} = \begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 1\ 0)}$$

$$\boxed{\textbf{Key property: } \text{If } V \perp W, \quad P_V + P_W = P_{V+W}}$$

Intuition:



Intuition:



$\Rightarrow$ To project into <u>any</u> space:
"Split" it into orthogonal lines,
and add up those projections

FACT: For a subspace $\mathcal{U} \subseteq \mathbb{R}^n$ with orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_k\}$,
orthogonal projection onto $\mathcal{U}$

$$P_{\mathcal{U}} = \boxed{\sum_{j=1}^{k} \underbrace{\vec{u}_j \vec{u}_j^T}_{n \times n \text{ matrix}}}$$

Proof: $P_{\mathcal{U}} = P_{\mathrm{span}\{u_1, \ldots, u_{k-1}\}} + \underbrace{P_{\mathrm{span}\{u_k\}}}_{= u_k u_k^T}$

Exercise: Compute the projection of $e_1 = (1, 0, 0, \ldots)$ onto a random 50-dimensional subspace of $\mathbb{R}^{100}$.

Answer:

```
n = 100;
d = 50;

x = zeros(n,1);
```

$\leftarrow$ the vector e_1

```
d = 50;

v = zeros(n,1);
v(1) = 1;
```

the vector e_1

```
A = randn(n, d);     # choose d random vectors in R^n,
                           #  with normally distributed coordinates
[Q,R] = qr(A,0);     # generate Q, whose columns form an
                           #  orthonormal basis for R(A),
                           #  the span of A's columns
```

we'll explain this function later

$Q(:,1), Q(:,2), \ldots, Q(:,d)$ form an orthonormal basis for $R(A)$

① First approach

```
projectedv = zeros(n,1);
for i = 1:d
    projectedv += Q(:,i) * (Q(:,i)' * v);
end for;
projectedv
```

build sum $\sum_{i=1}^{d} q_i q_i^T v$ one term at a time

② Second try: Note: $Q = \sum_{i=1}^{d} q_i e_i^T$ where $q_i = Q(:,i)$ ith column

$$\Rightarrow Q Q^T = \left(\sum_i q_i e_i^T\right)\left(\sum_j e_j q_j^T\right)$$

$$= \sum_{i,j} q_i \underbrace{(e_i^T e_j)}_{e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}} q_j^T$$

$$= \sum_{i=1}^{d} q_i q_i^T$$

```
projectedv2 = Q * (Q' * v);

sum(abs(projectedv2 - projectedv))
```

Trick: To check that two vectors are the same, add up the absolute values of the coord. differences

③ Check the answer:

```
# 1. Is projectedv in R(A)?
x = A\projectedv;          # Solve for a linear comb. of the columns of A
err = A*x - projectedv     #  that gives projectedv
sum(abs(err))
```

2. Is v-projectedv perpendicular to R(A)?
(v - projectedv)' * A

(Question: What is the expected squared length of the projection? projectedv' * projectedv)

Key Example: If $V = \mathbb{R}^n$, "resolution of the identity" is

$$\boxed{\sum_{j=1}^{n} \vec{v}_j \vec{v}_j^{\,T} = \underset{\text{identity matrix}}{I}}$$

This gives an easy derivation of the other facts:

Example: I forget, is

$$\vec{u} = \sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j \quad \text{or} \quad \vec{u} = \sum_j (\vec{u} \cdot \vec{v}_j) \vec{v}_j \ ?$$

over $\mathbb{C}$, they're different!

$$\vec{u} = I u$$
$$= \sum_j v_j v_j^T u = \sum_j v_j (v_j^T u) = \boxed{\sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j}$$

Example:

$$\|\vec{u}\|^2 = \vec{u}^T \vec{u}$$
$$= u^T I u$$
$$= u^T \sum_j v_j v_j^T u$$
$$= \sum_j (u^T v_j)(v_j^T u)$$

$$\Rightarrow \|u\|^2 = \sum_j |\vec{u} \cdot \vec{v}_j|^2$$

Any basis expansion can be done by inserting

$$I = \sum_j v_j v_j^T.$$

Even changes of basis:

$$B = \{\vec{b}_1, \ldots, \vec{b}_n\} \qquad C = \{\vec{c}_1, \ldots, \vec{c}_n\}$$

orthonormal sets

$$I = I \cdot I$$

$$= \sum_i c_i c_i^T \sum_j b_j b_j^T$$

$$= \sum_{i,j} \vec{c}_i (c_i \cdot b_j) b_j^T$$

⑧ Orthonormal bases simplify many arguments

- If you prove something for the standard basis, it most likely holds for any orthonormal basis.
  Intuition and arguments are usually basis independent.

<u>Example</u>: If $V \perp W$,

$$P_{V+W} = P_V + P_W .$$

Why? Work in an orthonormal basis

$$\{ \underbrace{\vec{b}_1, \ldots, \vec{b}_p}_{\text{basis for } V},\ \underbrace{\vec{b}_{p+1}, \ldots, \vec{b}_{p+q}}_{\text{basis for } W},\ \vec{b}_{q+p+1}, \ldots, \vec{b}_n \}$$

$$P_V = \begin{array}{c c} & \begin{array}{ccc} p & q & n-p-q \end{array} \\ \begin{array}{c} p \\ q \\ n-p-q \end{array} & \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \qquad P_W = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$P_V + P_W = \begin{array}{c c} & \begin{array}{cc} p+q & n-p-q \end{array} \\ \begin{array}{c} p+q \\ n-p-q \end{array} & \left(\begin{array}{cc|c} I_p & 0 & 0 \\ 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \quad \checkmark$$

<u>Example</u>: If $V \subseteq W$,

$$P_V P_W = P_W P_V = P_V .$$

Check this.

But how do we find orthonormal bases for spaces ?

(How does Matlab's qr function work?)

Projections? Ch. 5.13

1D projections

how to build larger projections?

general formula

from orthogonal projection $P+Q$, $PQ=0$

closest pt Thm. p. 535

projection to an affine subspace?

OUTLINE: Projections? Pseudoinverses?

Review

Orthonormal bases make life simpler

Interpret basis expansion in terms of projections

Gram-Schmidt procedure

Neely: orthogonality — 1 lecture

Gram-Schmidt, projections — 2 lectures

Outline: Orthogonal vectors

Orthogonality

Normed vectors

Gram-Schmidt

Note: • Having an orthogonal basis simplifies coordinate expansions greatly.

• But Gram-Schmidt often gives ugly results.

?: Other bases are nice?

Fourier transform?

full lecture? ~~FFT~~?

read p 299

dynamics equation

hard to understand/solve

but easy for waves!

$\Rightarrow$ decompose into sum of waves

linearity

examples, e.g., ocean waves

this is a spectral decomposition
tridiagonal matrix → diagonal