

Homework 5

Thursday, October 15, 2015 9:30 AM

Due in class 10/22.

- ① Using the Gram-Schmidt procedure, find an orthonormal basis for the span of

$$(1, 1, 1, -1), (2, -1, -1, 1) \text{ and } (-1, 2, 2, 1).$$

- ② a) Use the Gram-Schmidt procedure to find an orthonormal basis for the row space, column space, nullspace and left nullspace of

$$A = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & -2 & 6 \\ 5 & -10 & -5 & 15 \end{pmatrix}$$

- b) What is the projection of $(1, 1, 1)$ onto $N(A^T)$?

③ Orthonormal functions

For two functions $f, g: [0, 1] \rightarrow \mathbb{R}$, define their inner product to be

$$f \cdot g := \int_0^1 dx f(x)g(x).$$

- a) Notice that the monomials

$$1, x, x^2, x^3$$

form a basis for the set of polynomials of degree ≤ 3 (a vector space).

A polynomial can be represented by its vector of coordinates in this basis, e.g.,

$$\underbrace{1 + 2x - x^2}_{\text{polynomial}} \longleftrightarrow \underbrace{(1, 2, -1, 0)}_{\substack{\text{vector of coefficients} \\ \text{in basis } \{1, x, x^2, x^3\}}}$$

Write a Matlab function that takes

Input: Two polynomials f and g ,
given by their coefficient vectors

Output: $f \cdot g$

Check your answer! For example,

$\text{polynomialdot}([1, 2, 0, 0], [3, 4, 0, 0])$
should return 10.667, since

$$(1+2x) \cdot (3+4x) = \int_0^1 (1+2x)(3+4x) dx \\ = \frac{32}{3}$$

(This is different from the usual dot product)
 $(1, 2, 0, 0) \cdot (3, 4, 0, 0) = 1 \cdot 3 + 2 \cdot 4 = 11$.

b) Using your function from part a, verify that
the polynomials

$$v_0(x) = 1$$

$$v_1(x) = \sqrt{3}(2x-1)$$

$$v_2(x) = \sqrt{5}(6x^2-6x+1)$$

$$v_3(x) = \sqrt{7}(20x^3-30x^2+12x-1)$$

form an **orthonormal basis** for the set of polynomials
of degree ≤ 3 . In other words,

$$v_i \cdot v_j = \int_0^1 dx v_i(x) v_j(x) = \begin{cases} 1 & , \text{ if } i=j \\ 0 & , \text{ if } i \neq j. \end{cases}$$

Remark: Here's some Mathematica code that checks
orthonormality:

```
basis = {v0, v1, v2, v3} = {
  1,
  2 Sqrt[3] (x - 1/2),
  Sqrt[5] (6 x^2 - 6 x + 1),
  Sqrt[7] (20 x^3 - 30 x^2 + 12 x - 1)
};
Table[
  Integrate[basis[[i]] basis[[j]], {x, 0, 1}],
  {i, 1, 4}, {j, 1, 4}]
```

I want you to do it
in Matlab, using the vector
representation of these functions.

For example,

$$v_1 = (-\sqrt{3}, 2\sqrt{3}, 0, 0)$$

```

basis = {v0, v1, v2, v3} = {
  1,
  2  $\sqrt{3} \left(x - \frac{1}{2}\right)$ ,
   $\sqrt{5} (6x^2 - 6x + 1)$ ,
   $\sqrt{7} (20x^3 - 30x^2 + 12x - 1)$ 
};
Table[
  Integrate[basis[[i]] basis[[j]], {x, 0, 1}],
  {i, 1, 4}, {j, 1, 4}
] // MatrixForm

matrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$


```

I want you to do it in Matlab, using the vector representation of these functions. For example,

$$v_1 = (-\sqrt{3}, 2\sqrt{3}, 0, 0)$$

c) Start with the basis

(*) $1, x, x^2, x^3, x^4, \dots, x^9, x^{10}$
for polynomials of degree ≤ 10 .

Use the Gram-Schmidt method with your code from part @ to change this into an orthonormal basis (numerically, in Matlab).

Deliverable:

Please turn in your source code, as well as a printout of a matrix whose rows are your final orthonormal polynomials.

For example, when I did this working numerically in Matlab, the first four polynomials I got were the rows:

```

-----
octave:381> Q(1:4,:)
ans =

Columns 1 through 4:
      1      x      x2      x3
1 → 1.0000e+00  0.0000e+00  0.0000e+00  0.0000e+00
 $\sqrt{3}(2x-1)$  → -1.7321e+00  3.4641e+00  0.0000e+00  0.0000e+00
v2 → 2.2361e+00 -1.3416e+01  1.3416e+01  0.0000e+00
v3 → -2.6458e+00  3.1749e+01 -7.9373e+01  5.2915e+01

```

the other columns are 0s

These are the same four polynomials found above.

These are the same four polynomials found above.

Extra: Does your code still work for finding an orthonormal basis for polynomials of degree ≤ 40 ? Why or why not?

④ a) Is the matrix

$$U = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{pmatrix}$$

unitary? Why or why not?

b) Give a 2×2 matrix that maps

$$u = (1+i, i) \text{ to } 2u$$

$$\text{and } v = (1+i, -2i) \text{ to } -3v.$$

(Hint: Lecture 15 might help, together with part a)
Check your answer!!