Homework 6

Thursday, October 22, 2015 3:30 PM

Due in class Thursday 10/29/15.

1) Compute the singular-value decompositions of the following matrices:

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ (c) $C = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

What are the norms of the matrices? Please give exact answers (not just numerical).

2) Using the definition $\|A\| = \max\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$,

prove that for any invertible matrix A,

$$\|A\| = \frac{1}{\min\limits_{y:\|y\|=1} \|A^{-1}y\|}.$$

3) Recall the $n \times n$ matrix

$$A_n \equiv \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & 0 & & \ddots & \ddots & \ddots \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}.$$

(This matrix arose as a discretization of the second derivative, on an interval with periodic boundary conditions.) In this problem, you will solve for $\|A_n\|$, at least when n is even.

(a) Using Matlab or similar software, determine numerically the norms of

$$A_4 = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$A_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ \ldots \end{pmatrix}$

$$A_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

(You do not have to prove that your answers are correct.)

(b) For A_4 and A_6, find vectors achieving the norm, ie., nonzero vectors $\vec{u} \in \mathbb{R}^4$ and $\vec{v} \in \mathbb{R}^6$ satisfying

$$\|A_4 \vec{u}\| = \|A_4\| \cdot \|u\| \quad \text{and} \quad \|A_6 \vec{v}\| = \|A_6\| \cdot \|\vec{v}\|.$$

(Hint: Compute the SVD numerically, and inspect the results.)

(c) Generalize your answers to part (b), in order to prove a good lower bound on $\|A_n\|$, for n even.

(d) What is the norm of a permutation matrix, eg.,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{? Why?}$$

(e) Now prove an upper bound on $\|A_n\|$ that matches the lower bound you gave in part (c).

This finishes the calculation of $\|A_n\|$ (for n even).

(Hint: "Break A_n into pieces." That is, write A_n as the sum of three or four permutation matrices, possibly with weights, and then use

A_n as the sum of three or four permutation matrices, possibly with weights, and then use the triangle inequality $\|B+C\| \le \|B\| + \|C\|$.)

(f) [Optional] What about $\|A_n\|$ for n odd?

4) (a) <u>Without computing the norm exactly</u>, argue why

$$100 \le \left\| \begin{pmatrix} 1 & -100 \\ 0 & 1 \end{pmatrix} \right\| \le 101.$$

(b) What is the inverse of $A = \begin{pmatrix} 1 & -100 \\ 0 & 1 \end{pmatrix}$?

What is the condition number of A?

What is the condition number of A^{-1}?

(c) Find vectors $\vec{b} \in \mathbb{R}^2$ and $\vec{\delta} \in \mathbb{R}^2$ such that $\|\delta\|$ is "small" compared to $\|b\|$, and yet

$$\|A^{-1}(b+\delta) - A^{-1}b\|$$

is "large" compared to $\|A^{-1}b\|$. (Use your own judgment for what should count as small or large.)

(Hint: Compute the SVD and experiment a bit, using Matlab.)

5) (a) Let $H_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$.

Compute $b = H_3 x$ for $x = (1,1,1)$ and $x = (0, 6, -3.6)$.

A small change Δb produces a large change Δx.

(b) Compute numerically the largest and smallest singular values of the 7×7 "Hilbert matrix" H_7, a matrix whose (i,j) entry is $\frac{1}{i+j-1}$.

(Hint: Google the Matlab "hilb" command.)

(c) If $H_7 x = b$ with $\|b\| = 1$, how large can $\|x\|$ be?

If b has roundoff error less than 10^{-16} in norm, how large an error can this cause in x.