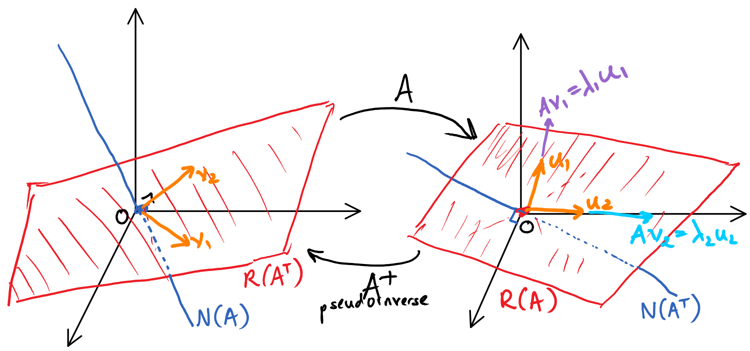
Lecture 19 : Introduction to eigenvectors

Thursday, October 29, 2015 9:30 AM

Admin:

INTRODUCTION TO EIGENVECTORS



SVD $\Rightarrow$ linear transformations are simple

- map rowspace to columnspace

  $\vec{v}_j \longmapsto \lambda_j \vec{u}_j$

  orthonormal basis to orthogonal basis

But some matrices are even simpler!

Example: Diagonal matrices

$$A = \begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}$$

$e_1 \mapsto 12 e_1$
$e_2 \mapsto 10 e_2$
$e_3 \mapsto -e_3$
$e_4 \mapsto 0 e_4$

just scales basis vectors without a rotation!

Why this is cool?

rank = 3, norm = 12, s.v.'s = 12, 10, 1, 0

$$A^{500} = \begin{pmatrix} 12^{500} & & & 0 \\ & 10^{500} & & \\ & & (-1)^{500} & \\ 0 & & & 0 \end{pmatrix}$$

easy to take powers!

Example: Diagonalizable matrices

$$A = \mathcal{U} \underbrace{\begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}}_{=D} \mathcal{U}^{-1}$$

where $\mathcal{U} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix} = \sum_j u_j e_j^T$

$\mathcal{U}\vec{e}_1 = \vec{u}_1$

$\Rightarrow \mathcal{U}^{-1}\vec{u}_1 = \vec{e}_1$, so $A\vec{u}_1 = 12\vec{u}_1$

A just scales each $\vec{u}_j$, without changing directions!

$$A^{500} = \mathcal{U} D \underbrace{\mathcal{U}^{-1} \cdot \mathcal{U}}_{I} D \underbrace{\mathcal{U}^{-1} \cdot \mathcal{U}}_{I} D \mathcal{U}^{-1} \cdot \cdots$$

$$= \mathcal{U} \cdot D^{500} \cdot \mathcal{U}^{-1}$$

$$= U \cdot D^{500} \cdot U^{-1}$$
$$= U \begin{pmatrix} 12^{500} & & & 0 \\ & 10^{500} & & \\ & & (-1)^{500} & \\ 0 & & & 0 \end{pmatrix} U^{-1} \quad \text{easy!}$$

Note: Directions of other vectors definitely do change,

e.g., $\underbrace{A(\vec{u}_1+\vec{u}_2)}_{= U\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} = \underbrace{12\vec{u}_1 + 10\vec{u}_2}_{= U\begin{pmatrix}12\\10\\0\\0\end{pmatrix}}$

Only the special vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$, and their **multiples**, have directions unchanged. These are called **eigenvectors**.

Again, $\text{rank}(A) = 3$.

If the $\{u_j\}$ are orthonormal,

then $\|A\| = 12$ and singular values are $12, 10, 1, 0$.

If the $\{u_j\}$ are not orthonormal, all bets are off!

Eg., $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$U \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U^{-1}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{norm} = \sqrt{2}$$

More examples of eigenvectors & of diagonalizable matrices

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad A\vec{v}_1 = \vec{v}_1$ $\quad$ eigenvector with eigenvalue 1

$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad A\vec{v}_2 = -\vec{v}_2$ $\quad$ eigenvalue -1

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

Generalization 1:

For any permutation matrix P,
the all-ones vector is a $+1$ eigenvalue e-vector.
(since permuting the coordinates leaves the all-ones vector alone).

Generalization 2:

For any square matrix whose rows all sum to 1 ("row-stochastic"), the all-ones vector is $+1$ ev ev.

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \qquad A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 + 1/2 \\ 1/3+1/3+1/3 \\ 1/4+1/4+1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \qquad A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix} 1/2+1/2 \\ 1/3+1/3+1/3 \\ 1/4+1/4+1/2 \end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

- $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$A\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \end{pmatrix} = \begin{pmatrix} e^{-i\theta} \\ ie^{-i\theta} \end{pmatrix} = e^{-i\theta}\begin{pmatrix}1\\i\end{pmatrix}$$

using $e^{i\theta} = \cos\theta + i\sin\theta$

$$A\begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix} c+is \\ s-ic \end{pmatrix} = \begin{pmatrix} c+is \\ -i(c+is) \end{pmatrix} = e^{i\theta}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

since $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ is unitary (col's are orthonormal)

so $U^{-1} = U^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

<u>Moral</u>: A real-valued matrix can have **complex** eigenvalues and eigenvectors.

In contrast, the SVD of a real matrix works with real left and right singular vectors, and singular values are always nonnegative reals.

Eg. A's singular values are 1 and 1.

- $A = \vec{u} \cdot \vec{v}^T$ for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$

What is the rank of A?

1, provided $\vec{u}, \vec{v} \neq \vec{0}$

What is the nullspace N(A)? $\dim N(A) = n-1$

$$A\vec{x} = \vec{u}\vec{v}^T\vec{x} = \vec{u}(\vec{v}\cdot\vec{x})$$

$$A\vec{x} = 0 \iff \vec{v}\cdot\vec{x} = 0$$

i.e. $x \perp v$

$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} = 0 \cdot \vec{x}$

<u>Moral</u>: Every vector in the nullspace of a matrix is an eigenvalue-0 eigenvector.

(Equivalently, the nullspace, if nonzero, is an eigenvalue-0 eigen<u>space</u>.)

Example:

$$A = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & \text{all-ones} & & 1 \\ 1 & & \text{matrix} & & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{n\times n} = \vec{u}\cdot\vec{u}^T \quad \text{for } \vec{u} = \frac{1}{\sqrt{n}}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

$$\Rightarrow A\vec{u} = uu^Tu$$
$$= \vec{u}(u\cdot u)$$

$= \|u\|^2 \vec{u}$

$= \vec{u}$

Moral: For the projection P_V onto a subspace V,

- V is a $+1$ eigenspace
  (for $x \in V$, $P_V x = x$)
- $V^\perp = N(P_V)$ is a 0 eigenspace
  (for $x \in V^\perp$, $P_V x = 0$)

- A slightly more complicated example:
  Let $A = \sum_i \lambda_i u_i v_i^T$ (an SVD)

  $\Rightarrow A^T A = \sum_i \lambda_i^2 v_i v_i^T$

  it just scales each $\vec{v}_i$ by λ_i^2, leaves direction unchanged

  $$\Rightarrow A^T A = \begin{pmatrix} | & | & | & | \\ | & v_i & | & | \\ | & | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} \text{—} \\ \text{—}\ v_i^T\ \text{—} \\ \text{—} \end{pmatrix}$$

- A matrix is diagonalizable $\iff$ there is a basis of eigenvectors (the columns of $\mathcal{U}$)

  $$A = \mathcal{U} \underset{\text{diagonal}}{D} \mathcal{U}^{-1}$$

From the examples so far, it seems that lots of matrices are diagonalizable.

But not all!!

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

good matrix to use in examples

SVD $A = e_1 \cdot e_2^T$
singular values $\lambda_1 = 1, \lambda_2 = 0$
rank 1, norm 1

Eigenvector $A\vec{e}_1 = \vec{0} = 0 \cdot \vec{e}_1$
eigenvalue 0

But there are no other (independent) eigenvectors!

$$A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \lambda x_2 = 0$$
$$\lambda x_1 = x_2$$

If $\lambda = 0$, then $x_2 = 0 \Rightarrow \vec{x} = x_1 \vec{e}_1$, which we already found

If $\lambda \neq 0$, again $x_2 = 0 \Rightarrow x_1 = 0 \Rightarrow \vec{x} = \vec{0}$ not an eigenvector

$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable!

It cannot be written $A = \mathcal{U} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathcal{U}^{-1}$

Alternative proof: Assume $A = \mathcal{U} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathcal{U}^{-1}$

Alternative proof: Assume $A = U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^{-1}$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = A^2 = (UDU^{-1})(UDU^{-1})$$
$$\begin{pmatrix}0&0\\0&0\end{pmatrix} = UD^2U^{-1}$$
$$\Rightarrow D^2 = U^{-1}\cdot 0 \cdot U = 0$$
$$\Rightarrow A = 0 \text{ , a contradiction } \times$$

"SPECTRAL THEORY" = THEORY OF EIGENVALUES & EIGENVECTORS

Spectral theory asks the following questions:

	Answers
① What matrices have an eigenvector?	$\begin{pmatrix}0&1\\0&0\end{pmatrix}$, ... all square matrices
② What matrices can be diagonalized? (That is, they have a full basis of eigenvectors.)	? diagonalizable matrices
③ What matrices can be diagonalized with an **orthogonal** basis of eigenvectors?	? "normal" matrices

Another natural question is when are the eigenvalues real? And, of course, how do you **find** eigenvalues & eigenvectors? We'll be studying these questions next.

But first, some applications:

Application 1: Solving linear recursions

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci sequence: $f_n = f_{n-1} + f$
$f_1 = f_2 = 1$

$$\begin{pmatrix}f_n\\ f_{n-1}\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}}_{A}\begin{pmatrix}f_{n-1}\\ f_{n-2}\end{pmatrix}$$

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}f_3\\f_2\end{pmatrix}$$
$$A^2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}f_4\\f_3\end{pmatrix}$$
$$\vdots$$
$$A^{500}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}f_{502}\\ f_{501}\end{pmatrix}$$

Unless you've seen this before you probably won't be able to guess the eigenvectors. Let's solve for them:

$$Ax = \lambda x \iff \begin{aligned} x_1 + x_2 &= \lambda x_1 \\ x_1 \quad &= \lambda x_2 \end{aligned}$$

(**Note**: We'll soon learn a better way of solving for eigenvectors & eigenvalues, but for a 2×2 matrix this won't be so bad.)

$$\Rightarrow (\lambda + 1) x_2 = \lambda^2 x_2$$

If $x_2 = 0$ then $\lambda = 1$, so $x_1 = 0 \longrightarrow$ no good!

$\Rightarrow x_2 \neq 0$, so we can divide out:

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Let $\tau \doteq \frac{1+\sqrt{5}}{2}$ the "golden ratio", $\tau^{-1} = -\frac{1-\sqrt{5}}{2}$

$\Rightarrow$

Eigenvalues	Eigenvectors
τ	$(\tau, 1)$
$-\tau^{-1}$	$(-\tau^{-1}, 1)$

these are perpendicular!

$$A = \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & -\tau^{-1} \end{pmatrix} \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\Rightarrow A^{500} = \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau^{500} & 0 \\ 0 & (-\tau^{-1})^{500} \end{pmatrix} \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix}^{-1}$$

Multiply by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to get out $\begin{pmatrix} f_{502} \\ f_{501} \end{pmatrix}$!

Notice: $\tau \approx 1.61$

$-\tau^{-1} = 1 - \tau \approx -.61$

$\Rightarrow \tau^{500} \gg |1-\tau|^{500}$

$\Rightarrow$ Asymptotically, the Fibonacci sequence grows like τ^n.

Application 2: Simple multi-variable differential equations

It is easy to solve

$$\frac{d}{dt} x(t) = x(t) \longrightarrow x(t) = \underset{\substack{\| \\ x(t=0)}}{C} e^t$$

What is the solution to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned} \quad ?$$

Still easy; $\ddot{y} = y \Rightarrow y = a \cdot e^{t} + b \cdot e^{-t}$

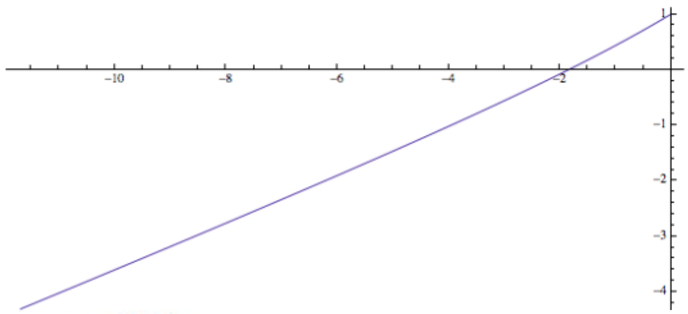
$$x = \dot{y} = a \cdot e^{t} - b \cdot e^{-t}$$

and use the initial conditions $x(0), y(0)$ to solve for a and b

What about these equations?

```
In[2]:= sol = NDSolve[{
        D[x[t], t] == 4 x[t] - 5 y[t],
        D[y[t], t] == 2 x[t] - 3 y[t],
        x[0] == 0,
        y[0] == 1
       }, {x, y}, {t, 0, 1}];
   ParametricPlot[{x[t], y[t]} /. sol, {t, 0, 1}]
```

Out[3]=




```
In[1]:= ?NDSolve
```

NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}] finds a numerical solution to the ordinary differential equations *eqns* for the function *u* with the independent variable *t* in the range t_{min} to t_{max}.
NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}, {*x*, x_{min}, x_{max}}] finds a numerical solution to the partial differential equations *eqns*.
NDSolve[*eqns*, {u_1, u_2, ...}, {*t*, t_{min}, t_{max}}] finds numerical solutions for the functions u_i. >>

Slightly harder to solve in your head! :)

"Rule": When you see multiple equations, try to vectorize them!

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases} \longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \dot{x} &= 4x - 5y \\ \dot{y} &= 2x - 3y \end{aligned} \longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

These both have the form

$$\frac{d}{dt} \vec{v}(t) = A \cdot \vec{v}(t)$$

If A can be diagonalized,

$$A = U D U^{-1}$$

$$\Rightarrow \frac{d}{dt} v = U D U^{-1} v$$

$$\Rightarrow U^{-1} \cdot \frac{d}{dt} v = \frac{d}{dt}(U^{-1} v) = D(U^{-1} v)$$

Change variables: $u = U^{-1} v$, $v = U u$,

$$\Rightarrow \frac{d}{dt} u = D u$$

$$\begin{pmatrix} \frac{d}{dt} u_1 \\ \frac{d}{dt} u_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{d}{dt}u_1 \\ \frac{d}{dt}u_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow u_1 = u_1(0)\, e^{\lambda_1 t}$$
$$u_2 = u_2(0)\, e^{\lambda_2 (t)}$$

$$\Rightarrow v = \mathcal{U} \cdot \begin{pmatrix} u_1(0)\, e^{\lambda_1 t} \\ u_2(0)\, e^{\lambda_2 t} \end{pmatrix}$$

The eigenvalues of A, λ_1 and λ_2, determine the exponents (growth rates).

We can actually simplify even further:

$$v(t) = \mathcal{U} \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} u(0)$$

$$= \mathcal{U} \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathcal{U}^{-1} v(0)$$

Functions of diagonalizable matrices:

Definition: If A is a diagonalizable matrix

$$A = \mathcal{U} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix} \mathcal{U}^{-1}$$

and $f : \mathbb{C} \to \mathbb{C}$ any function, define

$$f(A) = \mathcal{U} \begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ 0 & & \ddots & \\ & & & f(\lambda_n) \end{pmatrix} \mathcal{U}^{-1}.$$

Example: $f(x) = x^2$

$\Rightarrow f(A) = \mathcal{U} D^2 \mathcal{U}^{-1} = A^2$, as you'd expect ✓

Example: $f(x) = e^{tx}$ exponential

$$\Rightarrow f(A) = \mathcal{U} \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} \mathcal{U}^{-1}$$

So, $$v(t) = \underbrace{\mathcal{U} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathcal{U}^{-1}}_{e^{At}} v(0)$$

$\Rightarrow$ The solution to

$$\frac{d}{dt} v(t) = A\, v(t)$$

is $v(t) = e^{At} v(0)$.

This looks just like the single-variable case.

You can also show this using Taylor series:

$$e^{xt} = 1 + xt + \frac{(xt)^2}{2} + \cdots$$
$$= \sum_{j=0}^{\infty} \frac{(xt)^j}{j!}$$
and $$e^{At} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}$$ where $A^0 = \underset{\text{identity matrix}}{I}$.

More applications we'll consider later:

- Spectral graph partitioning
- Markov chains
- Google Pagerank

⋮

WHAT MATRICES HAVE AN EIGENVECTOR?

Theorem: Every (real or complex) square matrix has at least one eigenvector (over $\mathbb{C}$!).

Proof: Let A be an $n \times n$ matrix, real or complex.

Goal: Prove that there exists a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$ for some λ.

Step 1. Let $\vec{v}$ be any nonzero vector.

Consider the vectors

$$\vec{v},\ A\vec{v},\ A^2\vec{v},\ \ldots,\ A^n\vec{v}$$

Since there are $n+1$ vectors in an n-dimensional space, they must be linearly dependent.

(Note: This argument only works in finite dimensions.)

Say

$$\alpha_0 v + \alpha_1 A v + \alpha_2 A^2 v + \cdots + \alpha_n A^n v = 0$$

with not all α_j's 0.

Step 2. Thus

$$\left(\sum_{j=0}^{n} \alpha_j A^j\right)\vec{v} = 0$$

Factor the polynomial

$$p(x) = \sum_j \alpha_j x^j = \prod_j (x - \lambda_j)$$
$$\Rightarrow \left[\prod_j (A - \lambda_j I)\right]\vec{v} = \vec{0}$$

Step 3.

$\Rightarrow$ at least one of the matrix terms $A - \lambda_j I$ must be singular

(the product of two nonsingular matrices is nonsingular!)

✓ Done. □

Note: The first step can be "justified" by noticing that $\lim_{k\to\infty} \frac{A^k v}{\|A^k v\|}$ converges to an eigenvector with largest magnitude eigenvalue (though I don't want to prove this). So it makes sense to look at successive powers $A^k v$.

Note: There's a simple proof that $\mathbb{C}$ is algebraically closed in the appendix of Lang's "Linear Algebra."