Homework 7

Thursday, October 29, 2015 9:30 AM

<u>Note</u>: Of course, feel free to use Matlab/Mathematica when appropriate, but always show your work.

1) a) Show that the best least-squares fit to a set of measurements $y_1, \ldots, y_m$ by a *horizontal line* (a constant function $y = C$) is their average

$$C = \frac{y_1 + \cdots + y_m}{m}.$$

b) Show that the slope of the line that passes through the origin in $\Re^2$ and comes closest in the least squares sense to passing through the points $\{(x_1, y_1),\ (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $m = \sum_i x_i y_i / \sum_i x_i^2$.

2) An economist hypothesizes that the change (in dollars) in the price of a loaf of bread is primarily a linear combination of the change in the price of a bushel of wheat and the change in the minimum wage. That is, if B is the change in bread prices, W is the change in wheat prices, and M is the change in the minimum wage, then $B = \alpha W + \beta M$. Suppose that for three consecutive years the change in bread prices, wheat prices, and the minimum wage are as shown below.

	Year 1	Year 2	Year 3
B	+\$1	+\$1	+\$1
W	+\$1	+\$2	0\$
M	+\$1	0\$	−\$1

Use the theory of least squares to estimate the change in the price of bread in Year 4 if wheat prices and the minimum wage each fall by \$1. What is the R^2 value for your fit?

3) For $\mathbf{A} \in \Re^{m \times n}$ and $\mathbf{b} \in \Re^m$, prove that $\mathbf{x}_2$ is a least squares solution for $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{x}_2$ is part of a solution to the larger system

$$\begin{pmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \end{pmatrix} \quad \begin{pmatrix} \mathbf{b} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}.$$

4) Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p+q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

$R(A)$, $R(A^T)$, $N(A)$ or $N(A^T)$

5) Find the best straight-line fit (least squares) to the measurements

$$b=4 \text{ at } t=-2, \qquad b=3 \text{ at } t=-1,$$
$$b=1 \text{ at } t=0, \qquad b=0 \text{ at } t=2.$$

Then find the projection of $b=(4,3,1,0)$ onto the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

6) Find the best least-squares error **parabola** to the four points $(0,0)$, $(1,8)$, $(3,8)$, $(4,20)$. (Your answer should minimize the summed squared error in the y-coordinates) What is the R^2 value of your fit?

7) Is $(AB)^+ = B^+A^+$ always true for pseudoinverses? No! Give a counterexample.

8) Okuns "law", in economics, states that the annual change in gross domestic product (GDP) should relate to the annual change in the unemployment rate via an equation

$$\underbrace{\Delta G}_{\text{change in GDP}} = k - c \cdot \underbrace{\Delta U}_{\text{change in unemployment rate}}$$

I have uploaded two datasets, giving US GDP growth rates and unemployment rates. You can read them into Matlab using the csvread() function.

Find the least-squares best values for k and c.
Plot the data and your best-fit line.
Explain briefly what this means (one or two sentences).

(Note: The data gives the unemployment rates. You'll need to compute the changes in unemployment rates.)

9) Let A be an m×n real matrix, with singular-value decomposition

$$(*) \quad A = \sum_i \lambda_i \vec{u}_i \vec{v}_i^T$$

ⓐ Give singular-value decompositions for

AA^T and A^TA.

(Hint: Multiply Equation (*) by its transpose and simplify. Then justify why you have an SVD.)

ⓑ Using your answer to part ⓐ, and recalling that the rank of a matrix is the number of nonzero singular values, prove that:

- $\text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^TA)$
- $R(A) = R(AA^T)$
- $N(A) = N(A^TA)$

(Hint: There are lots of ways of proving the last two

(Hint: There are lots of ways of proving the last two statements. Maybe try expressing $R(A)$ in terms of the left singular vectors of A (the $\vec{u}_i$ vectors), and then do the same for AA^T.)

ⓒ Now assume that $m=n$ and A is invertible. In terms of the condition number κ of A, what is the condition number of A^TA?

ⓓ Give a singular-value decomposition for the $(m+n)\times(m+n)$ matrix

$$B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

(Hint: What is B applied to $(\vec{0}, \vec{v}_i) \in \mathbb{R}^{m+n}$?
How about B applied to $(\vec{u}_i, \vec{0}) \in \mathbb{R}^{m+n}$?)