Lecture 20: How to find eigenvectors

Tuesday, November 3, 2015 9:30 AM

Admin:

Reading: Meyer Ch. 6, 7.1-7.4
Strang Chs. 4-5

Recall:

$$A\vec{v} = \lambda\vec{v}$$

eigenvalue: λ; eigenvector: $\vec{v}$

A is diagonalizable $\iff$ $A = UDU^{-1}$

diagonal matrix $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

columns of U are eigenvectors
$A\vec{u}_i = \lambda_i \vec{u}_i$

$\Updownarrow$

there is a basis of eigenvectors of A

Example:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

evector e-value

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is <u>not</u> diagonalizable.

$A\vec{e}_1 = 0 \cdot \vec{e}_1$ but that's it!

Today: Every matrix has an eigenvector.
How to find eigenvalues and eigenvectors
- in theory (using determinants)
- in practice (using the "power method")

Application: Systems of differential equations

Problem: Solve

$$\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$$

subject to $\vec{v}(t=0) = \vec{v}_0$ initial conditions

Answer:

① In terms of eigenvalues/eigenvectors:

- Say $A\vec{u}_1 = \lambda_1\vec{u}_1$
  and $\vec{v}_0 = \vec{u}_1$

  $\Rightarrow \vec{v}(t)$ is <u>always</u> proportional to $\vec{u}_1$
  $\vec{v}(t) = e^{\lambda t}\vec{u}_1$

  (so the time derivative is $e^{\lambda t} \cdot \lambda_1 \vec{u}_1$
  $= A \cdot \vec{v}(t)$ ✓)

$\vec{v}(t) = e^{\lambda t}\vec{u}_1$

(so the time derivative is $e^{\lambda t}\cdot\lambda_1\vec{u}_1$
$= A\cdot\vec{v}(t)$ ✓)

- Say $A\vec{u}_1 = \lambda_1\vec{u}_1$
  $A\vec{u}_2 = \lambda_2\vec{u}_2$
  and $\vec{v}_0 = \vec{u}_1 - 2\vec{u}_2$

  $\Rightarrow \vec{v}(t) = e^{\lambda_1 t}\vec{u}_1 - e^{\lambda_2 t}\cdot 2\vec{u}_2$

  (by linearity:
  if $\frac{d}{dt}\vec{\omega}(t) = A\vec{\omega}(t)$
  and $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$,
  then $\frac{d}{dt}\left(\vec{\omega}(t)+\vec{x}(t)\right) = A\left(\vec{\omega}(t)+\vec{x}(t)\right)$)

- If A is diagonalizable, then its eigenvectors form a basis, so **any** $\vec{v}_0$ can be expressed as a linear combination

$$\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \cdots + \alpha_n\vec{u}_n$$

of eigenvectors — and then they evolve independently

$$\vec{v}(t) = \alpha_1 e^{\lambda_1 t}\vec{u}_1 + \cdots + \alpha_n e^{\lambda_n t}\vec{u}_n$$

(2) In terms of matrices:

$$v(t) = e^{At}\vec{v}_0$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

$$= \sum_{j=0}^{\infty}\frac{1}{j!}(At)^j$$

Proof:

- $v(0) = e^{A\cdot 0}\vec{v}_0$
  $= I\vec{v}_0$
  $= \vec{v}_0$ ✓

- $\frac{d}{dt}\left(e^{At}\vec{v}_0\right) = \left(\frac{d}{dt}e^{At}\right)\vec{v}_0$

$$= \left(\sum_{j=0}^{\infty}\frac{1}{j!}\cdot\frac{d}{dt}\left(A^j t^j\right)\right)\vec{v}_0$$

$$= \left(\sum_{j=1}^{\infty}\frac{1}{(j-1)!}A^j t^{j-1}\right)\vec{v}_0$$

$$= A\left(\sum_{j=0}^{\infty}\frac{1}{j!}(At)^j\right)\vec{v}_0$$

$$= A\vec{v}(t) \ ✓$$

Extension: Higher-order differential equations with constant coefficients.

To solve a 2^{nd}-order diff. eq. like

$$\ddot{x} = -\dot{y} + 2x$$
$$\ddot{y} = \dot{x} + \dot{y} - 3y$$

use the same trick we used for solving the Fibonacci sequence recursion:

$$\frac{d}{\cdot}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \cdot\end{pmatrix} = \begin{pmatrix}0 & -1 & 2 & 0\\ 1 & 1 & 0 & -3\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ x\end{pmatrix}$$

use the same trick we used for solving the Fibonacci sequence recursion:

$$\frac{d}{dt}\begin{pmatrix}\dot{x}\\ \dot{y}\\ x\\ y\end{pmatrix} = \begin{pmatrix}0 & -1 & 2 & 0\\ 1 & 1 & 0 & -3\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ x\\ y\end{pmatrix}$$

Then continue as before.

What Matrices Have an Eigenvector?

Theorem: Every (real or complex) square matrix has at least one eigenvector (over $\mathbb{C}$!).

Proof: Let A be an $n \times n$ matrix, real or complex.

Goal: Prove that there exists a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$ for some λ.

Step 1. Let $\vec{v}$ be any nonzero vector.

Consider the vectors

$$\vec{v},\ A\vec{v},\ A^2\vec{v},\ \ldots,\ A^n\vec{v}$$

Since there are $n+1$ vectors in an n-dimensional space, they must be linearly dependent.

(**Note**: This argument only works in finite dimensions.)

Say

$$\alpha_0 v + \alpha_1 A v + \alpha_2 A^2 v + \cdots + \alpha_n A^n v = 0$$

with not all α_j's 0.

Step 2. Thus

$$\left(\sum_{j=0}^{n} \alpha_j A^j\right)\vec{v} = 0$$

Factor the polynomial

$$p(x) = \sum_j \alpha_j x^j = \prod_j (x - \lambda_j)$$

$$\Rightarrow \left[\prod_j (A - \lambda_j I)\right]\vec{v} = 0$$

Step 3.

$\Rightarrow$ at least one of the matrix terms $A - \lambda_j I$ must be singular

(the product of two nonsingular matrices is nonsingular!)

✓ Done. $\square$

Note: The first step can be "justified" by noticing that $\lim_{k\to\infty} \frac{A^k v}{\|A^k v\|}$ converges to an eigenvector with largest magnitude eigenvalue (though I don't want to prove this). So it makes sense to look at successive powers $A^k v$.

DETERMINANTS (in brief)

Definition: The determinant of an $n \times n$ square matrix A is

$$\sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \underbrace{\text{sign}(\sigma)}_{= (-1)^{\#\text{ of transpositions required to get to } \sigma}} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

Examples:

- $n=1$: $A = (A_{11})$

  $\det(A) = A_{11}$

- $n=2$: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Permutations	Sign	
$1 \mapsto 1, 2 \mapsto 2$	$+1$	(0 transpositions)
$1 \leftrightarrow 2$	-1	

$$\Rightarrow \boxed{\det A = ((+1) \cdot a_{11} a_{22}) + ((-1) \cdot a_{12} a_{21}) = a_{11} a_{22} - a_{12} a_{21}}$$

- $n=3$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Permutation σ	Sign(σ)	$\prod_{i=1}^{3} a_{i,\sigma(i)}$
identity	$+1$	$a_{11} a_{22} a_{33}$
$2 \leftrightarrow 3$	-1	$a_{11} a_{23} a_{32}$
$1 \leftrightarrow 2$	-1	$a_{12} a_{21} a_{33}$
$1 \to 2 \to 3 \to 1$	$+1$	$a_{12} a_{23} a_{31}$
$1 \to 3 \to 2 \to 1$	$+1$	$a_{13} a_{21} a_{32}$
$1 \leftrightarrow 3$	-1	$a_{13} a_{22} a_{31}$

since $1 \to 2 \to 3 \to 1$ = $1 \leftrightarrow 2$ followed by $1 \leftrightarrow 3$ (two transpositions)

add these up to get the determinant!

Observe:

$$\det(A) = a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

- General n: the same recursion works

  $a_{11} \cdot \det(n-1 \times n-1$ submatrix missing 1st row, 1st col$)$

  $- a_{12} \cdot \det(n-1 \times n-1$ submatrix missing row 1, col 2$)$

  $+ a_{13} \cdot \ldots$

  $\vdots$

$+(-1)^n a_{1n} \cdot \det(\text{submatrix missing row } 1, \text{col } n)$

[You can actually do this across any row or down any column; note that $\det(A) = \det(A^T)$.]

Problem: There are $n! \approx \left(\frac{n}{e}\right)^n$ permutations of n elements; although

$$\det(A) = \sum_{\substack{n\text{-element}\\ \text{permutations } \sigma}} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

defines the determinant, that's **not** how it is actually computed!

Compare: The permanent of a square matrix A is

$$\text{perm}(A) = \sum_{\substack{n\text{-element}\\ \text{permutations } \sigma}} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

without the sign! Computing the permanent is NP-hard.

More examples:

- $\text{Det}\begin{pmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{pmatrix} = abcd$ product of diagonal entries
  diagonal matrix

- $\text{Det}\begin{pmatrix} a & e & f & g \\ 0 & b & h & i \\ 0 & 0 & c & j \\ 0 & 0 & 0 & d \end{pmatrix} = abcd$ still!
  upper triangular matrix

true for permanent, too

false for permanent!

Cool fact (that I won't prove or use):

$$\text{Det}(AB) = \text{Det}(A) \cdot \text{Det}(B)$$

Example:

```
>> n = 50;
>> A = randn(n,n);
>> B = randn(n,n);
>> det(A*B)

ans =

   3.1827e+61

>> det(A) * det(B)

ans =

   3.1827e+61
```

Note:

```
>> factorial(50)

ans =

   3.0414e+64
```

— how did Matlab compute $\det(A), \det(B)$?

Answer: It takes the LU decomposition:

Answer: It takes the LU decomposition:

$$A = \underbrace{P}_{\text{permutation matrix}} \cdot \underbrace{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}}_{\text{lower triangular}} \cdot \underbrace{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}_{\text{upper triangular}}$$

$$\Rightarrow \det(A) = \det(P) \cdot \det(L) \cdot \det(U).$$

More on this later...

Corollary: If A is diagonalizable,

$$A = U \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} U^{-1},$$

then $\underbrace{\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n}_{\text{product of the eigenvalues}}$.

Proof:

$$\det(A) = \det(UDU^{-1})$$
$$= \det(U) \cdot \det(D) \cdot \det(U^{-1})$$

But $U \cdot U^{-1} = I$

$$\Rightarrow \det(U) \cdot \det(U^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(A) = \det(D) = \lambda_1 \cdots \lambda_n \quad \checkmark \square$$

Key fact (that we will prove):

$$\det(A) \neq 0 \iff A \text{ is invertible}$$

~ "nonsingular"
~ inverse exists
~ $\operatorname{rank}(A) = n$

$$\det(A) = 0 \iff A \text{ is not invertible}$$

~ $\operatorname{rank}(A) < n$

COROLLARY: HOW TO FIND EIGENVECTORS:

Observe: λ is an eigenvalue of A

$$\Updownarrow$$

$A\vec{v} = \lambda \vec{v}$ for a nonzero vector $\vec{v}$

$$\Updownarrow$$

$(A - \lambda I)\vec{v} = 0$ for $\vec{v} \neq 0$

$$\Updownarrow$$

$N(A - \lambda I) \neq \{0\}$.

$$\Updownarrow$$

$A - \lambda I$ is singular!

$$\Updownarrow$$

$$\det(A - \lambda I) = 0$$

$$\boxed{\det(A-\lambda I)=0 \iff \lambda \text{ is an eigenvalue of } A}$$

Recipe:
1. Compute $\det(A-\lambda I)$
2. Find all roots, $\det(A-\lambda I)=0$
3. For each λ, corresponding eigenspace is $N(A-\lambda I)$ — find it with Gaussian elimination!

Problem: Calculate the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Answer: A "

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$
$$= (3-\lambda)(3-\lambda)-1$$
$$= \lambda^2 - 6\lambda + 8$$
$$= (\lambda-4)(\lambda-2) = 0$$

$\Rightarrow$ Two eigenvalues, $\lambda_1 = 4$ and $\lambda_2 = 2$

Note: In general, if A is an $n\times n$ matrix, $\det(A-\lambda I)$ will be a degree-n polynomial in λ. Why?

Note: $\det A = 8 = \lambda_1 \cdot \lambda_2$

— this is generally true!

$\text{trace}(A)$ = sum of diagonal elements
$= 6$
$= \lambda_1 + \lambda_2$

— also generally true!

For 2×2 matrices, these two equations are enough to solve for λ_1 and λ_2.

Next, the eigenspaces:

- $N(A-\lambda_1 I) = N\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right)$
$= \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

- $N(A-\lambda_2 I) = N\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$
$= \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓

Done: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

e-vectors e-values

Observe: The eigenvectors of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ are the same as those for $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{B}$.

If $B\vec{v} = \lambda\vec{v}$, $(B+3I)\vec{v} = (\lambda+3)\vec{v}$

— But the eigenvalues of the sum of two matrices will <u>not</u> generally be the sums of the eigenvalues, unless they have the same eigenvectors.

Problem: Calculate the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Answer: First, give it a name (always a good idea!).

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now we solve for λ in

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 + 1 \end{aligned}$$

$\Rightarrow \lambda \in \{i, -i\}$

They add to 0 since $\mathrm{Trace}(A) = 0$

Their product is $1 = \det(A)$. ✓

To calculate the respective eigenvectors, we could solve for $N(A \pm iI)$. But it is easier just to go directly:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = i\begin{pmatrix} 1 \\ y \end{pmatrix}$$

$\overset{\parallel}{\begin{pmatrix} -y \\ 1 \end{pmatrix}}$ $\Downarrow y = -i$ ✓

Similarly,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix} \checkmark$$

Problem: What are the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \;?$$

Answer: A "

$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}$ is upper triangular

$\Rightarrow$ determinant is product of diagonal entries

$\Rightarrow$ eigenvalue 3 (with multiplicity 2)

Eigenvectors: Solve

$$0 = (A - 3I)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

and we may scale so $x_1 = 1$

$\Rightarrow$ eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$

is the <u>only</u> eigenvector!

(The matrix is not diagonalizable)

<u>Problem</u>: What are the eigenvalues of

$$\begin{pmatrix} 3 & 5 & 8 & 13 \\ & 2 & 3 & 5 \\ & & 1 & 2 \\ 0 & & & 1 \end{pmatrix} ?$$

Answer: Just as above, since the matrix is upper triangular, its eigenvalues are the diagonal elements: 3, 2 and 1 (with multiplicity 2). ✓

<u>Moral</u>: Triangular matrices are easy!

<u>Question</u>: Why does an $n \times n$ matrix have at most n distinct eigenvalues?

Answer 1:

Because $\det(A - \lambda I)$ is a degree n polynomial in λ, it always has ≥ 1 and $\leq n$ roots.

Answer 2:

Assume there are $n+1$ <u>different</u> eigenvalues:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$
$$A\vec{v}_{n+1} = \lambda_{n+1}\vec{v}_{n+1}$$

$\Rightarrow$ e-vectors cannot be linearly independent

If, say, the first n are lin. indep., then

$$\vec{v}_{n+1} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$

for some constants $\alpha_1, \ldots, \alpha_n$

Apply A to both sides

$$A\vec{v}_{n+1} = \lambda_{n+1}\vec{v}_{n+1}$$

$$\alpha_1\lambda_1\vec{v}_1 + \cdots + \alpha_n\lambda_n\vec{v}_n \quad = \quad \alpha_1\lambda_{n+1}\vec{v}_1 + \cdots + \alpha_n\lambda_{n+1}\vec{v}_n$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \underbrace{(\lambda_j - \lambda_{n+1})}_{\text{nonzero!}} \vec{v}_j = 0$$

$\Rightarrow$ contradicts that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent.

<u>Note</u>: Dotting the i's, this argument actually proves

<u>Lemma</u>: Any set of eigenvectors for a matrix, corresponding to <u>distinct</u> eigenvalues, is linearly independent.

Recall: λ is an eigenvalue of A

$\Updownarrow$

$A\vec{v} = \lambda\vec{v}$ for a nonzero vector $\vec{v}$

$\Updownarrow$

$(A-\lambda I)\vec{v} = 0$ for $\vec{v} \neq 0$

$\Updownarrow$

$N(A-\lambda I) \neq \{0\}$.

$\Updownarrow$

$A-\lambda I$ is singular!

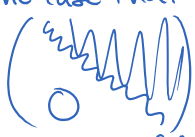
$\Updownarrow$

$\det(A-\lambda I) = 0$

we still need to prove this step!

Proof that $\det(A) \neq 0 \iff A$ is nonsingular:

① This is true in the case that A is upper triangular.

$A =$ 

A is nonsingular $\iff \text{rank}(A) = n$

$\iff$ all diagonal entries are nonzero

$\iff \det A$ (product of diagonal entries) nonzero

② To reduce to the upper-triangular case, use Gaussian elimination! (Sorry.)

Gaussian elimination has one basic operation:

Add a multiple of one row to another row.

How does this affect the determinant?

Lemma 1: If a row is repeated $\Rightarrow \det(A) = 0$.

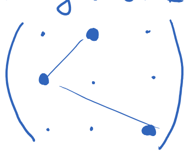
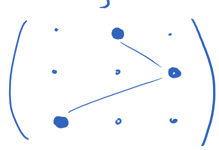
Example: $\det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ba = 0$ ✓

```
>> n = 10;
>> A = randn(n-1,n);
>> A = [A; A(5,:)];
>> det(A)

ans =

  -1.6929e-14
```

Why?

$$\det(A) = \sum_{\substack{\text{permutations} \\ \sigma}} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

If rows j and k are the same, then every term appears twice!

$a_{12}a_{21}a_{33}$ $\quad$ $a_{12}a_{23}a_{31}$

$a_{12}a_{21}a_{33}$ $\qquad$ $a_{12}a_{23}a_{31}$

the same if row 2 = row 3

But the permutations for matching terms differ by one transposition — so have opposite signs, and cancel out. ✓ $\square$

Observation 2: The determinant function is definitely not linear,

$$\text{e.g., } \det(10 \cdot A) = 10^n \cdot \det(A).$$

But each summand in

$$\det(A) = \sum_{\text{permutations } \sigma} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

involves exactly one entry from any given row of A.

$$\Rightarrow \det\begin{pmatrix} 5a_{11} & 5a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 5 \cdot \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and, more importantly,

$$\det\begin{pmatrix} \text{— row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } n \text{ —} \end{pmatrix} + \det\begin{pmatrix} \text{— a different row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } n \text{ —} \end{pmatrix}$$

$$= \det\begin{pmatrix} \text{row 1 + different row 1} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } n \text{ —} \end{pmatrix}$$

Corollary: Adding one row to another does not change the determinant (by Obs. 2 and Lemma 1).

$$\det\begin{pmatrix} \text{row 1 + row 5} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } n \text{ —} \end{pmatrix} = \det\begin{pmatrix} \text{row 1} \\ \text{row 2} \\ \vdots \\ \text{row } n \end{pmatrix} + \underbrace{\cancel{\det\begin{pmatrix} \text{row 5} \\ \text{row 2} \\ \text{row 3} \\ \vdots \\ \text{row } n \end{pmatrix}}}_{=0}$$

since row 5 is repeated

$\Rightarrow$ Applying Gaussian elimination to A does not change whether $\det A = 0$ or $\det A \neq 0$.

After Gaussian elimination, $\det A' = 0 \Leftrightarrow \text{rank}(A') < n$.
Hence, before Gaussian elimination, $\det A = 0 \Leftrightarrow \text{rank}(A) < n$.

✓ $\square$

Summary of Eigenvalues & Eigenvectors

To find the eigenvalues of an $n \times n$ matrix A, find the roots λ where $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ of degree n

$\Rightarrow$ it can be factored

$$\det(A - \lambda I) = (\lambda - \lambda_1)^{\alpha_1} \cdot (\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_k)^{\alpha_k}$$

with $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$.

(hence $k \leq n$)

The eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$.

For each distinct eigenvalue λ_i,

$\dim N(A-\lambda_i I) \geq 1$ (since it is an eigenvalue)

and $\dim N(A-\lambda_i I) \leq \alpha_i$

$\Rightarrow$ There are at least k independent eigenvectors.

There are n independent eigenvectors (i.e., A is diagonalizable) if and only if

$\dim N(A-\lambda_i I) = \alpha_i$ for every $i = 1, \ldots, k$

Why is $\dim N(A-\lambda_i I) \leq \alpha_i$?

Proof: We want to show that $\operatorname{rank}(A-\lambda_i I) \geq n-\alpha_i$; the claim $\dim N(A-\lambda_i I) \leq \alpha_i$ then follows by the Rank-Nullity theorem.

<u>**Useful claim**</u>: If $\mathcal{U}$ is invertible, then the eigenvalues of A are the same as those of $\mathcal{U}A\mathcal{U}^{-1}$.

Proof: If $A\vec{v} = \lambda\vec{v}$, then

$$(\mathcal{U}A\mathcal{U}^{-1})(\mathcal{U}\vec{v}) = \mathcal{U}A\vec{v} = \lambda(\mathcal{U}\vec{v}) \checkmark$$

Now apply Gaussian elimination to A; assuming no row interchanges are required, this gives

$$A = \left(L\right)\left(\mathcal{U}\right)$$

where L has 1s along the diagonal and $\mathcal{U}$ has each λ_i on its diagonal α_i times. (Think about it....)

Thus we can also write

$$A = L\,\mathcal{U}'L^{-1};$$

this means applying the same operations to the <u>columns</u> of $\mathcal{U}$ as were done to the <u>rows</u> of A in Gaussian elimination. It leaves the diagonal entries unchanged.

But then

$$\begin{aligned}\operatorname{rank}(A-\lambda_i I) &= \operatorname{rank}(\mathcal{U}'-\lambda_i I)\\ &\geq \text{\# of nonzero entries along the diagonal, since } \mathcal{U}'-\lambda_i I \text{ is upper triangular}\\ &= n-\alpha_i. \checkmark \quad \square\end{aligned}$$

<u>Note</u>: The recipe

① Compute $p(\lambda) = \det(A-\lambda I)$

② Find its factors $\lambda_1, \ldots, \lambda_k \Rightarrow$ eigenvalues of A

③ Compute nullspaces $N(A-\lambda_i I) \Rightarrow$ eigenspaces

(2) Find its factors $\lambda_1, \ldots, \lambda_k \Rightarrow$ eigenvalues of A
(3) Compute nullspaces $N(A - \lambda_i I) \Rightarrow$ eigenspaces

works in theory, and for 2×2, 3×3 matrices.

But it quickly becomes impractical. Writing down $p(\lambda)$, and then finding its roots, is very time-consuming.

Next time we'll learn a faster way...

In fact, a common way of solving for the roots of a polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

is to compute the eigenvalues of a matrix!

For

$$A = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & \text{O} & -a_1 \\ & 1 & \ddots & & \vdots \\ & \text{O} & & & \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

$$\det(A - \lambda I) = p(\lambda).$$

$$\begin{vmatrix} \lambda & a_0 \\ -1 & \lambda + a_1 \end{vmatrix} = \lambda^2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & 0 & a_0 \\ -1 & \lambda & a_1 \\ 0 & -1 & \lambda + a_2 \end{vmatrix} = \lambda^3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & & \text{O} & a_0 \\ -1 & \lambda & & a_1 \\ & -1 & \lambda & a_2 \\ \text{O} & & -1 & \lambda + a_3 \end{vmatrix} = \lambda^4 + \lambda^3 a_3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$\vdots$