Homework 8

Thursday, November 5, 2015 9:30 AM

FIRST, SOME BASIC FACTS:

1) Prove that if $\mathcal{U}$ is invertible, then the eigenvalues of

$$A$$

are the same as the eigenvalues of

$$\mathcal{U} A \mathcal{U}^{-1}.$$

How are the eigenvectors of A related to the eigenvectors of $\mathcal{U} A \mathcal{U}^{-1}$?

2) If A is an $n \times n$ matrix, explain why

$$\det(\alpha A) = \alpha^n \cdot \det(A)$$

for all scalars α.

3) What are the eigenvalues and corresponding eigenvectors for the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & 0 & \\ & & \lambda_3 & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{pmatrix} ?$$

4) For a polynomial $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$, and a square matrix A, define $p(A)$ to be the matrix

$$p(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_k A^k.$$

Show that if $\vec{v}$ is an eigenvector of A, with corresponding eigenvalue λ, then $\vec{v}$ is also an eigenvector of $p(A)$, with corresponding eigenvalue $p(\lambda)$.

5) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors of A associated with the same eigenvalue λ, explain why every nonzero linear combination

$$x = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_k \vec{v}_k$$

is also an eigenvector for A with eigenvalue λ.

NEXT, SOME CALCULATIONS:

6) (a) What are the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} 0 & -3 & -2 \\ 2 & 5 & 2 \\ -2 & -3 & 0 \end{pmatrix} ?$$

What is its determinant?

Find a nonsingular matrix U such that $U^{-1}AU$ is a diagonal matrix.

Do this 3 ways:

- By hand, using $\mathrm{Det}(A - \lambda I) = 0, \ldots$ Show your work, and check your answers.
- In Matlab, using the power method

  In Matlab, using the built-in functions

(b) What are the eigenvalues and corresponding eigenvectors for

$$B = \begin{pmatrix} 1 & -3 & -2 \\ 2 & 6 & 2 \\ -2 & -3 & 1 \end{pmatrix} ?$$

Hint: Use your result from part ⓐ. If this takes more than a line or two to solve, you're doing it wrong...

7) ⓐ What are the eigenvalues and corresponding eigenspaces of the $n \times n$ matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & & & 1 \\ 1 & 1 & 2 & 1 & \cdots & & & 1 \\ \vdots & \vdots & 1 & 2 & \ddots & & 1 & \vdots \\ & & & \ddots & & & 2 & 1 \\ 1 & 1 & \cdots & & & 1 & 1 & 2 \end{pmatrix}$$

(2's on the diagonal, 1's everywhere else)? What is its determinant?

ⓑ What is the determinant of the $n \times n$ matrix

$$\begin{pmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 1 & & 1 & 1-n \end{pmatrix}$$

(1-n along the diagonal, 1's off the diagonal)? Hint: What are the eigenvalues?

8) ⓐ If $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, find A^{100} by diagonalizing A.

ⓑ Diagonalize $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ as $B = UDU^{-1}$ for some diagonal matrix D. Then use the diagonalization to prove that

$$B^k = \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix}.$$

9) The higher-order differential equation $y'' + y = 0$ can be written as a first-order system by introducing the derivative y' as another unknown:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}.$$

If this is $\frac{d\vec{v}}{dt} = A\vec{v}$, what is the 2×2 matrix A?

Find its eigenvalues and eigenvectors, and compute the solution that starts from $y(0)=2$, $y'(0)=0$.