Homework 9

Thursday, November 12, 2015 9:30 AM

1) [Review] What is the pseudoinverse of the $n \times 1$ matrix

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} ?$$

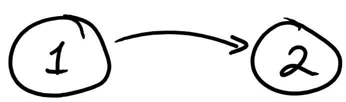
2) Prove that a matrix that is upper- or lower-triangular is normal $\iff$ it is diagonal.

3) Prove that the product of stochastic matrices is also stochastic.

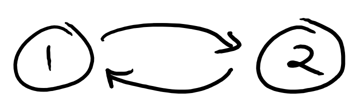
4) For the PageRank framework covered in class, prove that the importance of a page that nobody links to is $2/n$.

5) A directed graph, eg., of web hyperlinks, is "strongly connected" if any vertex (webpage) can be reached from any other vertex (webpage) by following directed edges.

Examples:



connected, not strongly connected



strongly connected

In this exercise, you will prove:

Theorem: Let A be the link matrix, as defined in class, for a strongly connected graph. Then A's eigenvalue-one eigenspace has dimension one.

(**Recall**: In class, we proved this if $A_{ij} > 0$ for all i, j.)

ⓐ Note that page i can be reached from page j in one step if and only if $A_{ij} > 0$ (since $A_{ij} > 0$ means there is a link from page j to page i).

Show that $(A^2)_{ij} > 0$ if and only if page i can be reached from page j in exactly two steps.

(**Hint**: Use the definition $(A^2)_{ij} = \sum_k A_{ik} A_{kj}$, together with the fact that all A_{ij} are ≥ 0.)

ⓑ Show more generally that $(A^p)_{ij} > 0$ if and only if page i can be reached from page j in <u>exactly</u> p steps.

ⓒ Argue that $(I + A + A^2 + \cdots + A^p)_{ij} > 0$ if and only if page i can be reached from page j in p or <u>fewer</u> steps.

ⓓ Explain why every entry of $I + A + A^2 + \cdots + A^{n-1}$ is > 0 when the graph is strongly connected.

ⓔ Use ⓓ and problem ③ to show that

$$B = \frac{1}{n}\left(I + A + A^2 + \cdots + A^{n-1}\right)$$

is stochastic and entry-wise > 0. Conclude that

$$\dim N(B - I) = 1.$$

ⓕ Show that if $A\vec{v} = \vec{v}$, then $B\vec{v} = \vec{v}$, and conclude that $\dim N(A - I) = 1$. This finishes the proof! $\square$

6) The following code creates a 100×100 symmetric matrix, each entry of which is uniformly random from $[0, 1]$:

```
rng(1)  % seed the random number generator
        % this way everyone will get the same answer!
n = 100;
A = rand(n,n)
A = diag(diag(A)) + triu(A,1) + triu(A,1)'
    % (triu(A,1) extracts the portion of A strictly above the diagonal,
    % so triu(A,1)' mirrors that below the diagonal)
```

Run this code in Matlab or Octave to initialize A.

(a) Use the power method to compute the eigenvector of A corresponding to the largest-magnitude eigenvalue. Verify that you have indeed computed an eigenvector. What is the eigenvalue? Show your work.

(After you are done, you can check your answer by calling eigs(A,1), but please solve this problem, and the parts (b), (c), (d) below, without using the eig() or eigs() functions.)

(b) A is a nonsingular matrix. Now use the power method to compute the smallest-magnitude eigenvalue and a corresponding eigenvalue. Of course, show your work.

(Hint: If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1} — so the smallest-magnitude eigenvalues of A correspond to the largest-magnitude eigenvalues of A^{-1}.)

Please do this without computing the inverse matrix A^{-1}. You don't need to compute the LU decomposition of A, but why might doing so speed up your calculations?

(c) Finally, use the power method to find the 2nd & 3rd smallest magnitude eigenvalues and corresponding eigenvectors.

smallest magnitude eigenvalues and corresponding eigen-vectors.

(Note: You can check your answer by calling

eigs(A, 3, 'sm')

The 'sm' option tells Matlab to look for the smallest-magnitude eigenvalues. Once again, though, don't use this in your solution. I want you to use the power method!)