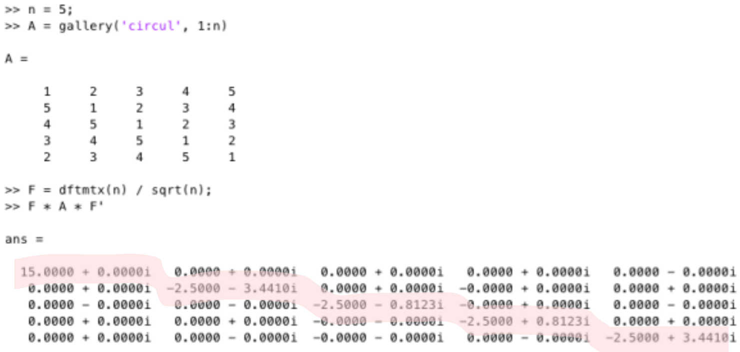
Lecture 25: Quantum physics

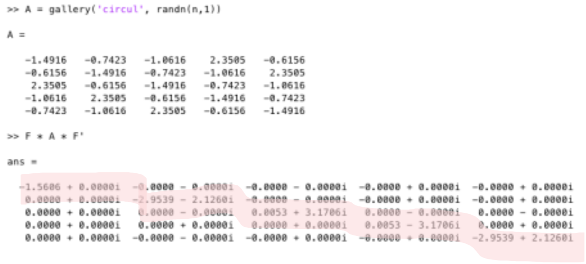
Tuesday, November 19, 2013 9:30 AM

Admin: Homework 10 problem 2 — don't use Matlab!

Aside: The Fourier transform diagonalizes circulant matrices.



```
>> n = 5;
>> A = gallery('circul', 1:n)

A =

     1     2     3     4     5
     5     1     2     3     4
     4     5     1     2     3
     3     4     5     1     2
     2     3     4     5     1

>> F = dftmtx(n) / sqrt(n);
>> F * A * F'

ans =

  15.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i
   0.0000 + 0.0000i  -2.5000 - 3.4410i   0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 - 0.0000i   0.0000 - 0.0000i  -2.5000 - 0.8123i  -0.0000 + 0.0000i   0.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i  -2.5000 + 0.8123i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i   0.0000 - 0.0000i  -2.5000 + 3.4410i
```



```
>> A = gallery('circul', randn(n,1))

A =

   -1.4916   -0.7423   -1.0616    2.3505   -0.6156
   -0.6156   -1.4916   -0.7423   -1.0616    2.3505
    2.3505   -0.6156   -1.4916   -0.7423   -1.0616
   -1.0616    2.3505   -0.6156   -1.4916   -0.7423
   -0.7423   -1.0616    2.3505   -0.6156   -1.4916

>> F * A * F'

ans =

  -1.5606 + 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i
   0.0000 + 0.0000i  -2.9539 - 2.1260i  -0.0000 - 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0053 + 3.1706i   0.0000 - 0.0000i   0.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0053 - 3.1706i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i  -2.9539 + 2.1260i
```

Exercise: Prove this!

Today: Quantum mechanics

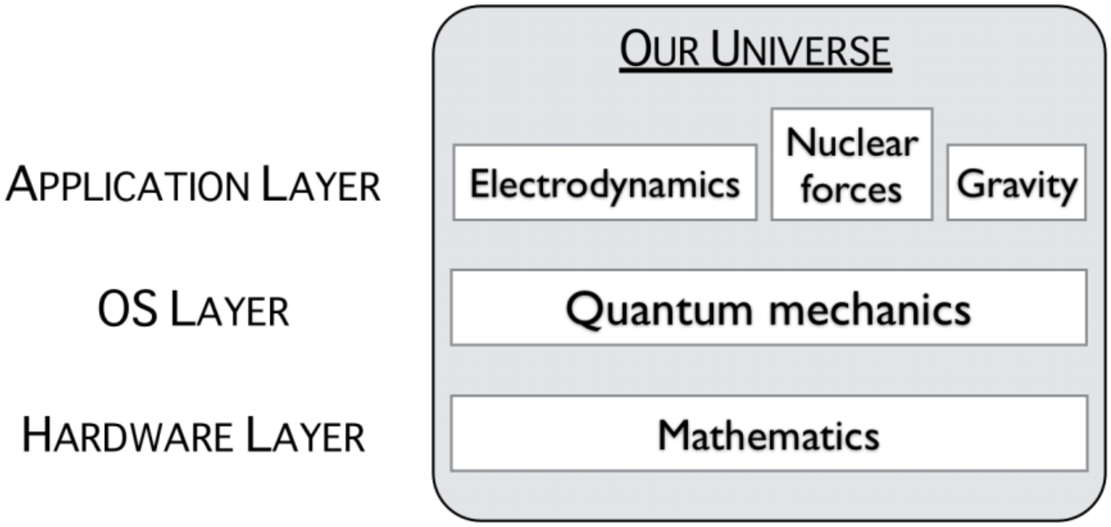
Morals: Matrix exponentiation & eigenvectors
to solve differential equations

Relationship between symmetric/Hermitian
and orthogonal/unitary matrices

Commuting matrices and time-dependent
differential equations

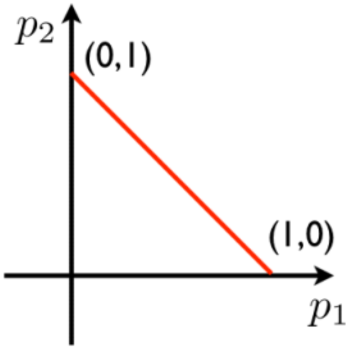
Tensor products to build bigger vector spaces

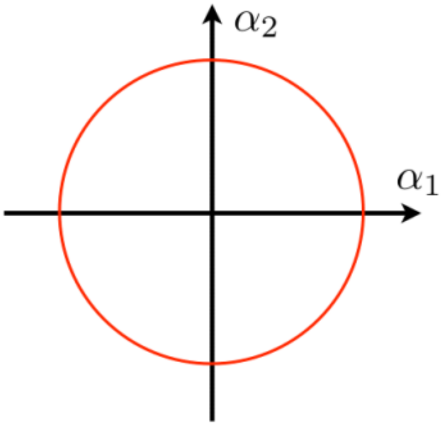
WHAT IS QUANTUM MECHANICS?



Quantum mechanics generalizes probability theory:

Probability theory	Quantum mechanics
$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$	$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathbb{C}^2$
$p_j \geq 0,\ p_1 + p_2 = 1$	$\lvert\alpha_1\rvert^2 + \lvert\alpha_2\rvert^2 = 1$

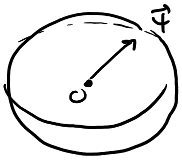




Definition: QUANTUM MECHANICS

System ⟷ Vector space (over $\mathbb{C}$)

State of the system ⟷ Vector $\vec{\psi}$ with $\|\vec{\psi}\| = 1$



Energy/"Hamiltonian" ⟷ Hermitian matrix H

(its real eigenvalues are discrete energy levels)

$$\text{energy}(\vec{\psi}) = \vec{\psi}^{\dagger} H \vec{\psi}$$

Observe: The "energy" is real:

$$(\psi^{\dagger} H \psi)^{\dagger} = \psi^{\dagger} H^{\dagger} \psi$$

$$= \psi^{\dagger} H \psi \quad \text{since } H^{\dagger} = H. ✓$$

Dynamics ⟶ "Schrödinger's equation"

$$\frac{d}{dt} \vec{\psi}(t) = -i H \vec{\psi}(t)$$

Solution: $\vec{\psi}(t) = e^{-iHt} \vec{\psi}(0)$

Observe: H can be diagonalized as

$$H = U D U^{\dagger}$$

$$\underset{\text{unitary}}{\wr} \quad = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \leftarrow \text{real e-values!}$$

$$\Rightarrow e^{-iHt} = U \begin{pmatrix} e^{-i\lambda_1 t} & & & 0 \\ & e^{-i\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{-i\lambda_n t} \end{pmatrix} U^\dagger$$

① If $\vec{\psi}$ is an eigenvector, $H\vec{\psi} = \lambda\vec{\psi}$,

$$e^{-iHt}\vec{\psi} = e^{-i\lambda t}\vec{\psi}$$

$\Rightarrow$ energy eigenvectors are <u>static</u>, except for accumulating a phase

② $$(e^{-iHt})(e^{-iHt})^\dagger = e^{-iHt}e^{iHt}$$

$$= U \begin{pmatrix} e^{-i\lambda_1 t} & & \\ & \ddots & \\ & & e^{-i\lambda_n t} \end{pmatrix} \begin{pmatrix} e^{i\lambda_1 t} & & \\ & \ddots & \\ & & e^{i\lambda_n t} \end{pmatrix} U^\dagger$$

$$= UU^\dagger$$

$$= I$$

$\Rightarrow$ time evolution is unitary (it preserves vector lengths)

Theorem:

Matrix A is unitary $\Leftrightarrow$ $A = e^{iH}$ for some Hermitian matrix H.

Proof:

We just showed that e^{iH} is unitary for any Hermitian matrix H.

For the converse statement: Assume A is unitary. Diagonalize it

$$A = U \begin{pmatrix} e^{i\theta_1} & & & 0 \\ & e^{i\theta_2} & & \\ & & \ddots & \\ 0 & & & e^{i\theta_n} \end{pmatrix} U^\dagger \quad \text{for real angles } \theta_1, \ldots, \theta_n$$

$$= \exp\Big(i \cdot \underbrace{U \begin{pmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{pmatrix} U^\dagger}_{\text{a Hermitian matrix!}}\Big) \quad ✓ \qquad \square$$

Example: Vector space $\mathbb{C}^2$

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{real symmetric} \Rightarrow \text{Hermitian}$$

Q: If $\vec{\psi}(\overset{\text{time}}{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is $\vec{\psi}(\overset{\text{time}}{\pi/2})$?

Answer:

E-values	E-vectors
$\mathcal{O}_1 = 1$	$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
$\mathcal{O}_2 = -1$	$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$\Rightarrow H = \mathcal{O}_1 v_1 v_1^\dagger + \mathcal{O}_2 v_2 v_2^\dagger$$

$$\Rightarrow e^{-iHt} = e^{-i\mathcal{O}_1 t} v_1 v_1^\dagger + e^{-i\mathcal{O}_2 t} v_2 v_2^\dagger$$

$$= \frac{e^{-it}}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{e^{it}}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\cos t & -i\cdot\sin t\\ -i\cdot\sin t & \cos t\end{pmatrix}$$

$$\Rightarrow \text{If } \psi(0) = \begin{pmatrix}1\\0\end{pmatrix}, \quad \psi(t) = \begin{pmatrix}\cos t\\ -i\sin t\end{pmatrix}$$

$$\Rightarrow \psi(\pi/2) = \begin{pmatrix}0\\-i\end{pmatrix}$$

Remark: Many physical systems are infinite dimensional. To track an electron through space, you need a dimension for every point in $\mathbb{R}^3$!

$\Rightarrow$ Exact diagonalization of Hamiltonians is often impossible, and we need to rely on approximations and perturbation theory.

Perturbation theory: If we understand the spectrum of A, and $\|B\| \ll \|A\|$, what can we say about the spectrum of $A+B$?

SIMULTANEOUS DIAGONALIZABILITY OF MATRICES

"Normal" matrices are nice because they can be unitarily diagonalized.

$$A \text{ normal} \Rightarrow \text{in some orthonormal basis, } A = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{pmatrix}$$

This makes matrix operations easy.

$$A^{100} = \begin{pmatrix}\lambda_1^{100} & & & 0\\ & \lambda_2^{100} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{100}\end{pmatrix} \text{ in this basis}$$

$$A \quad = \begin{pmatrix} & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^{100} \end{pmatrix}$$ in this basis

$$e^A = \begin{pmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{pmatrix}$$

⇒ trivial to solve differential equations $\frac{d}{dt}\vec{v} = A\vec{v}$

$A\vec{x} = \vec{b}$ is solved by

$$\vec{x} = \sum_{j=1}^{n} \frac{(v_j \cdot b)}{\lambda_j} \cdot \vec{v}_j \qquad \left(\text{since } \vec{b} = \sum_j (v_j \cdot b)\vec{v}_j\right)$$

$\text{Rank}(A) = \#$ of nonzero λ_j

$\text{Det}(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

etc...

But what if you want to work with **two** matrices, at the same time?

Examples: (from physics)

Total energy (a Hermitian matrix) = Kinetic energy + Potential energy

Energy of two-particle system = Energy of 1st particle + Energy of 2nd particle + Interaction energy

⋮

PROBLEM:

Say that matrices A and B are both unitarily diagonalizable.

When are they diagonalizable using the same unitary?

Equivalently, when do they share a basis of eigenvectors?

Definition: Matrices A and B **commute** if

$$AB = BA.$$

Examples:

- Any two $n \times n$ diagonal matrices commute:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\Rightarrow AB = \begin{pmatrix} 1\cdot 3 & 0 \\ & \end{pmatrix}, \quad BA = \begin{pmatrix} 3\cdot 1 & 0 \\ & \end{pmatrix} \checkmark$$

- All matrices commute with the identity:

$$I \cdot A = A \cdot I = A \quad \checkmark$$

- Not all matrices commute:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Theorem: For normal matrices A and B,

there is an orthonormal basis of e-vectors for <u>both</u> A and B $\iff$ $AB = BA$.

Proof:

<u>Easy direction $\Rightarrow$</u>:

Assume that A and B are diagonalized by the same unitary U:

$$A = U D_A U^{-1}$$
$$B = U D_B U^{-1},$$

with D_A and D_B diagonal.

$$\Rightarrow AB = (U D_A U^{-1})(U D_B U^{-1})$$
$$= U D_A D_B U^{-1}$$
$$BA = U D_B D_A U^{-1}$$

Since D_A and D_B are diagonal, $D_A D_B = D_B D_A$, so $AB = BA$. $\checkmark$

<u>Hard direction $\Leftarrow$</u>:

Assume that A and B commute: $AB = BA$.

Work in a basis so that A is diagonal:

$$A = \left(\begin{array}{c|c|c|c} \lambda_1 \ddots \lambda_1 & & & \\ \hline & \lambda_2 \ddots \lambda_2 & & \\ \hline & & \ddots & \\ \hline & & & \lambda_k \ddots \lambda_k \end{array}\right)$$

with $\lambda_1, \ldots, \lambda_k$ distinct e-values

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & \\ B_{21} & B_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & B_{kk} \end{pmatrix}$$

written in the same basis

$$\Rightarrow AB = \begin{pmatrix} \lambda_1 B_{11} & \lambda_1 B_{12} & \cdots & \lambda_1 B_{1k} \\ \lambda_2 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_2 B_{2k} \end{pmatrix}$$

$$\Rightarrow AB = \begin{pmatrix} \lambda_1 B_{11} & \lambda_1 B_{12} & \cdots & \lambda_1 B_{1k} \\ \lambda_2 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_2 B_{2k} \\ \vdots & \vdots & & \vdots \\ \lambda_k B_{k1} & \lambda_k B_{k2} & \cdots & \lambda_k B_{kk} \end{pmatrix}$$

$$BA = \begin{pmatrix} \lambda_1 B_{11} & \lambda_2 B_{12} & \cdots & \lambda_k B_{1k} \\ \lambda_1 B_{21} & \lambda_2 B_{22} & \cdots & \lambda_k B_{2k} \\ \vdots & \vdots & & \vdots \\ \lambda_1 B_{k1} & \lambda_2 B_{k2} & \cdots & \lambda_k B_{kk} \end{pmatrix}$$

Since $AB = BA$,

$\lambda_i B_{ij} = \lambda_j B_{ij}$ for all i, j

$\Rightarrow$ If $i \neq j$ then $B_{ij} = (0)$

Thus in this basis,

$$A = \begin{pmatrix} \begin{array}{c|c|c|c} \lambda_1 \ddots \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & \lambda_2 \ddots \lambda_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \lambda_k \ddots \lambda_k \end{array} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{kk} \end{pmatrix}$$

Since B is normal, $BB^\dagger = B^\dagger B = \begin{pmatrix} B_{11}B_{11}^\dagger & & 0 \\ & \ddots & \\ 0 & & B_{kk}B_{kk}^\dagger \end{pmatrix}$

(note that this is true in any basis, as $(UBU^\dagger)(UBU^\dagger)^\dagger = UBB^\dagger U^\dagger$)

$\Rightarrow$ each block B_{ii} is normal, too.

Choose a basis for each coordinate block so as to diagonalize the B_{ii}'s.

This does not affect A! (Since A is proportional to the identity on each block, $\lambda_i I$, and the identity I is the same in any basis.)

$\Rightarrow$ Now A and B are both diagonal. ✓ $\square$

Remark: The same theorem also extends to three or more matrices, with pairwise commutation, eg.,

$$AB = BA, \quad AC = CA, \quad BC = CB.$$

Theorem: If A, B and C are normal matrices that pairwise commute, then there is a basis $\vec{v}_1, \ldots, \vec{v}_n$ of simultaneous eigenvectors for all three:

$$A\vec{v}_j = \lambda_j^A \vec{v}_j \ , \ B\vec{v}_j = \lambda_j^B \vec{v}_j \ , \ C\vec{v}_j = \lambda_j^C \vec{v}_j \, .$$

In this basis all three matrices are diagonal:

$$A = \begin{pmatrix} \lambda_1^A & & & 0 \\ & \lambda_2^A & & \\ & & \ddots & \\ 0 & & & \lambda_n^A \end{pmatrix}, \ B = \begin{pmatrix} \lambda_1^B & & & 0 \\ & \lambda_2^B & & \\ & & \ddots & \\ 0 & & & \lambda_n^B \end{pmatrix}, \ C = \begin{pmatrix} \lambda_1^C & & 0 \\ & \ddots & \\ 0 & & \lambda_n^C \end{pmatrix}.$$

Example: Matrix exponentials

For real or complex numbers a and b,

$$e^{a+b} = e^a e^b$$

(eg. $e^{r_1 t_1} e^{r_2 t_2} = e^{r_1 t_1 + r_2 t_2}$)

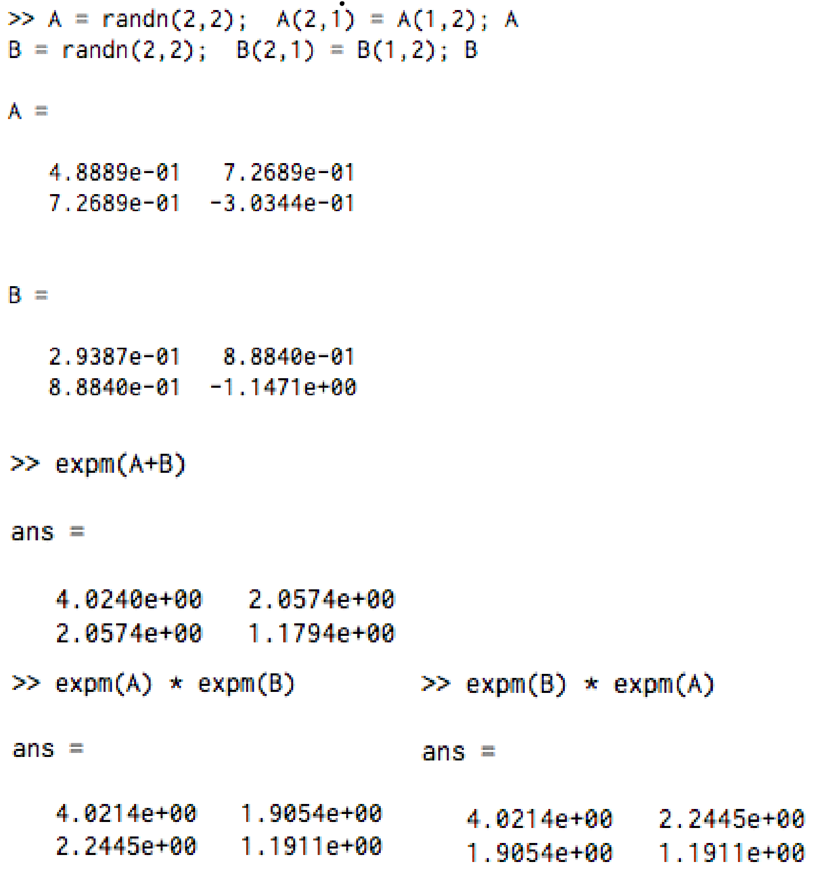
What about matrices?

If $A = U_A \underset{\text{diagonal}}{D_A} U_A^{-1}$,

$$\exp(A) = e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j = U_A \cdot e^{D_A} \cdot U_A^{-1}.$$

Similarly if $B = U_B D_B U_B^{-1}$.

What is e^{A+B}?



```
>> A = randn(2,2);  A(2,1) = A(1,2); A
B = randn(2,2);  B(2,1) = B(1,2); B

A =

   4.8889e-01   7.2689e-01
   7.2689e-01  -3.0344e-01


B =

   2.9387e-01   8.8840e-01
   8.8840e-01  -1.1471e+00


>> expm(A+B)

ans =

   4.0240e+00   2.0574e+00
   2.0574e+00   1.1794e+00
```

```
>> expm(A) * expm(B)

ans =

   4.0214e+00   1.9054e+00
   2.2445e+00   1.1911e+00
```

```
>> expm(B) * expm(A)

ans =

   4.0214e+00   2.2445e+00
   1.9054e+00   1.1911e+00
```

Problem: $AB - BA \neq 0$

```
>> A * B - B * A

ans =

            0  -3.4349e-01
   3.4349e-01            0
```

$$\exp(A+B) = I + A + B + \underbrace{\frac{(A+B)^2}{2}} + \cdots$$

$$= \frac{A^2+AB+BA+B^2}{2}$$

$$\exp(A)\exp(B) = \left(I+A+\frac{A^2}{2}+\cdots\right)\left(I+B+\frac{B^2}{2}+\cdots\right)$$

$$= I + A + B + \left(AB + \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + \right)\cdots$$

But if A and B commute, then $BA = AB$, so all the A's can be pulled to the left, and

$$e^{A+B} = e^A e^B = e^B e^A \quad \text{(if they commute!)}$$

Application: Time-dependent differential equations:

$$\frac{dy}{dt} = f(t)\,y(t)$$

$$\Rightarrow y(t) = y(0)\cdot\exp\left(\int_0^t f(s)\,ds\right).$$

But this is false for matrices

$$\frac{d\vec{y}}{dt} = F(t)\,\vec{y}(t)$$

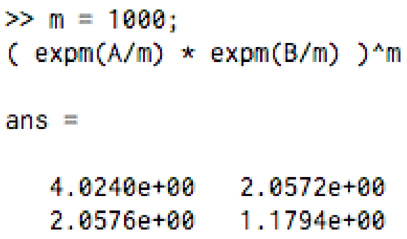
$$\not\Rightarrow \vec{y}(t) = \exp\left(\int_0^t F(s)\,ds\right)\vec{y}(0)$$

unless all matrices $F(t)$ commute with each other!

If they don't commute, then there isn't a clean solution, and you'll have to solve the differential equation numerically, using a sufficiently fine discretization of time.

```
>> expm(A+B)

ans =

   4.0240e+00   2.0574e+00
   2.0574e+00   1.1794e+00
```

```
>> m = 100;
( expm(A/m) * expm(B/m) )^m

ans =

   4.0240e+00   2.0552e+00
   2.0596e+00   1.1794e+00
```



```
>> m = 1000;
( expm(A/m) * expm(B/m) )^m

ans =

   4.0240e+00   2.0572e+00
   2.0576e+00   1.1794e+00
```

TRIVIA: Is this robust?

What if $\|AB-BA\| < \delta$?

Are they nearly simultaneously diagonalizable?

P. R. Rosenthal: Are almost commuting matrices near commuting pairs?. AMS Monthly **76**, 925 (1969).

first asked in the 1950s

H. Lin: Almost commuting self-adjoint matrices and applications. Fields. Inst. Commun. **13**, 193 (1995).

Theorem: There exists a function $\varepsilon = \varepsilon(\delta)$ such that, for any $n \times n$ Hermitian matrices A and B, with

$$\|AB-BA\| < \delta,$$

there exist commuting matrices A' and B' with

$$\|A-A'\| < \varepsilon(\delta) \quad , \|B-B'\| < \varepsilon(\delta).$$

Note: ε does not depend on n!

Theorem: [Hastings, 2008]

This holds for $\varepsilon(\delta) \sim \delta^{1/5}$.

Remark: These theorems are all false for almost-commuting triples of Hermitian matrices, and pairs of unitary matrices.

TENSOR PRODUCTS

Starting with a vector space V, you can get a smaller vector space by taking a subspace.

How to get a bigger vector space?

Method 1: "Direct sums"

Inputs: vector spaces $V \cong \mathbb{R}^m$, $W \cong \mathbb{R}^n$

Output: $V \oplus W \cong \mathbb{R}^{m+n}$

If $\vec{v} \in V$, $\vec{w} \in W$, then $(\vec{v}, \vec{w}) \in V \oplus W$

Examples: $\mathbb{R}^2 = \mathbb{R}^1 \oplus \mathbb{R}^1$

$$\begin{aligned} \mathbb{R}^3 &= \mathbb{R}^1 \oplus \mathbb{R}^2 \\ &= \mathbb{R}^2 \oplus \mathbb{R}^1 \\ &= \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1 \end{aligned}$$

If A is an $m \times m$ matrix acting on V, then

$$\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$ acts on the V part of $V \oplus W$

Matrices like $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$ act separately on V and W

Method 2: "Tensor products"

Inputs: $V \cong \mathbb{R}^m$, $W \cong \mathbb{R}^n$

Output: $V \otimes W \cong \mathbb{R}^{m \cdot n}$

Claim: You already know tensor products!

Proof by example: Consider two coins...

Coin 1

H	T
p	$1-p$

Coin 2

H	T
q	$1-q$

Coins 1 and 2 together

HH	HT	TH	TT
pq	$p(1-q)$	$(1-p)q$	$(1-p)(1-q)$

Thus $\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^4$

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \otimes \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} pq \\ p(1-q) \\ (1-p)q \\ (1-p)(1-q) \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 \cdot q_1 \\ p_1 \cdot q_2 \\ p_2 \cdot q_1 \\ p_2 \cdot q_2 \end{pmatrix} \in \mathbb{R}^4$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 \cdot q_1 \\ p_1 \cdot q_2 \\ p_2 \cdot q_1 \\ p_2 \cdot q_2 \\ p_3 \cdot q_1 \\ p_3 \cdot q_2 \end{pmatrix} \in \mathbb{R}^{6=3\times 2}$$

Note: Not all vectors in $\mathbb{R}^4$ are tensor products of vectors in $\mathbb{R}^2$ and $\mathbb{R}^2$.

HH	HT	TH	TT
$\frac{1}{2}$	0	0	$\frac{1}{2}$

coins glued together!

$$\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \neq \begin{pmatrix} p\,q \\ p\,(1-q) \\ (1-p)\,q \\ (1-p)(1-q) \end{pmatrix} \text{ for any } p, q$$

Independent events ⟷ Tensor-product distributions

Correlated events ⟷ Not a tensor product

Tensor products of operators

How to randomize a coin:

multiply by $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

since $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

How to randomize the 2nd of 2 coins:

multiply by $\begin{pmatrix} 1/2 & 1/2 & & \\ 1/2 & 1/2 & \multicolumn{2}{c}{0} \\ & & 1/2 & 1/2 \\ \multicolumn{2}{c}{0} & 1/2 & 1/2 \end{pmatrix}$

M

since $M \begin{pmatrix} a \\ b \\ c \\ d = 1-a-b-c \end{pmatrix} \begin{array}{l} HH \\ HT \\ TH \\ TT \end{array} = \begin{pmatrix} (a+b)/2 \\ (a+b)/2 \\ (c+d)/2 \\ (c+d)/2 \end{pmatrix}$

And the first coin?

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{array}{l} HH \\ HT \\ TH \\ TT \end{array} \begin{pmatrix} (a+c)/2 \\ (b+d)/2 \\ (a+c)/2 \\ (b+d)/2 \end{pmatrix}$$

Definition: $\Big(a_{11} B \quad a_{12} B \quad \ldots$

Definition:

$$A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \\ \vdots & & \ddots \end{pmatrix}$$

$$\Rightarrow I\otimes\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix} & 0 \\ 0 & \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\otimes I = \begin{pmatrix} I/2 & I/2 \\ I/2 & I/2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad ✓$$

Quantum entanglement

Definition: A quantum state in $\mathbb{C}^m\otimes\mathbb{C}^n$ is **entangled** if it is not a tensor product state.

Observe:

$$\begin{aligned} &1 \text{ coin} \longrightarrow \mathbb{C}^2 \\ &2 \text{ coins} \longrightarrow \mathbb{C}^2\otimes\mathbb{C}^2 = \mathbb{C}^4 \\ &3 \text{ coins} \longrightarrow \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2 = \mathbb{C}^8 \\ &\quad\vdots \\ &n \text{ coins} \longrightarrow (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n} \end{aligned}$$

The dimension grows exponentially!

Having to keep track of an exponentially long vector is why quantum systems are hard to simulate

— and also why quantum computers are potentially exponentially faster than standard (classical) computers!

Example: Bell inequality violation

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Editor's summary

NATURE | LETTER

Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres

B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau & R. Hanson

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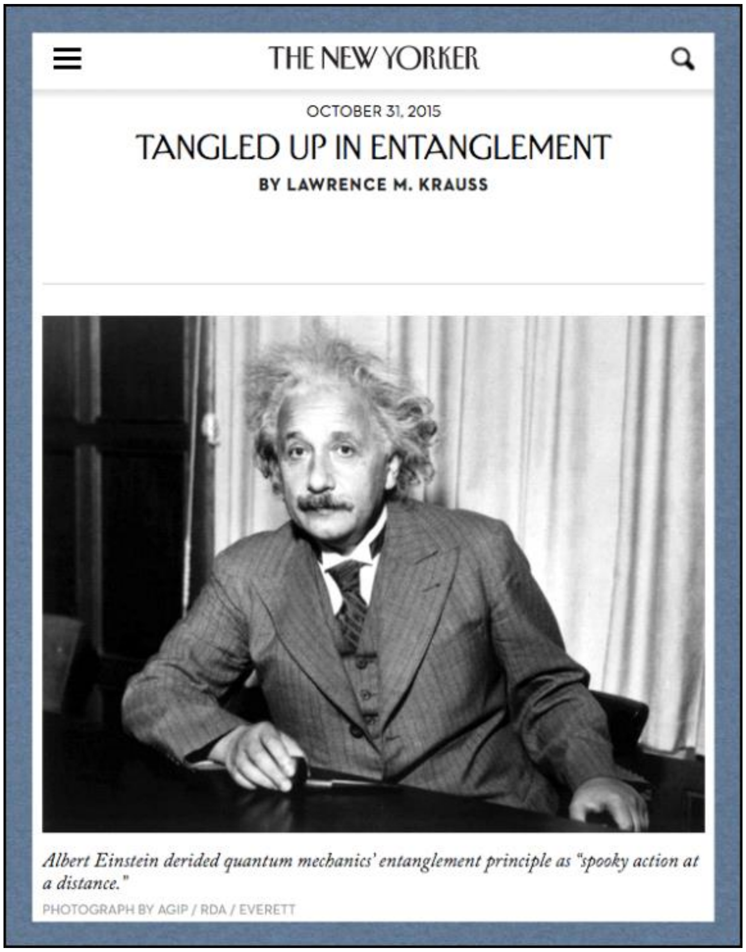


OCTOBER 31, 2015

TANGLED UP IN ENTANGLEMENT

BY LAWRENCE M. KRAUSS

Albert Einstein derided quantum mechanics' entanglement principle as "spooky action at a distance."

PHOTOGRAPH BY AGIP / RDA / EVERETT

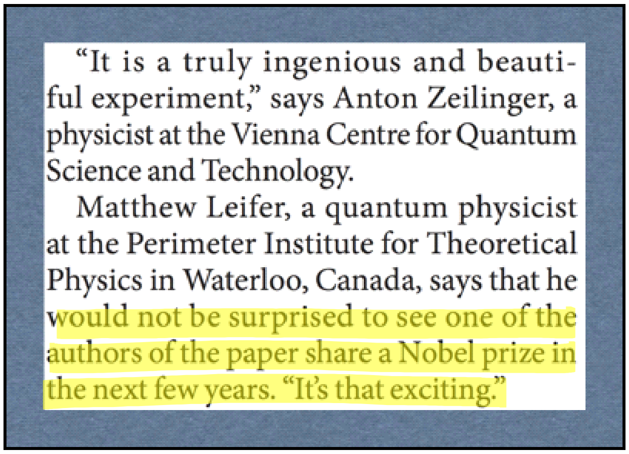
SCIENCE

Sorry, Einstein. Quantum Study Suggests 'Spooky Action' Is Real.

By JOHN MARKOFF OCT. 21, 2015

Part of the laboratory setup for an experiment at Delft University of Technology, in which two diamonds were set 1.3 kilometers apart, entangled and then shared information.

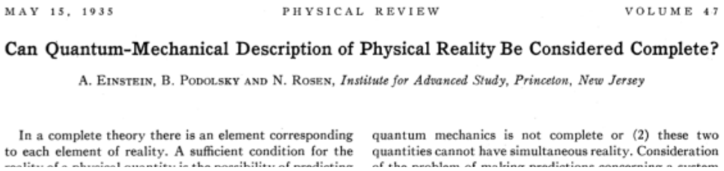
Frank Auperle/Delft University of Technology



"It is a truly ingenious and beautiful experiment," says Anton Zeilinger, a physicist at the Vienna Centre for Quantum Science and Technology.

Matthew Leifer, a quantum physicist at the Perimeter Institute for Theoretical Physics in Waterloo, Canada, says that he would not be surprised to see one of the authors of the paper share a Nobel prize in the next few years. "It's that exciting."

Einstein-Podolsky-Rosen (EPR) test



MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

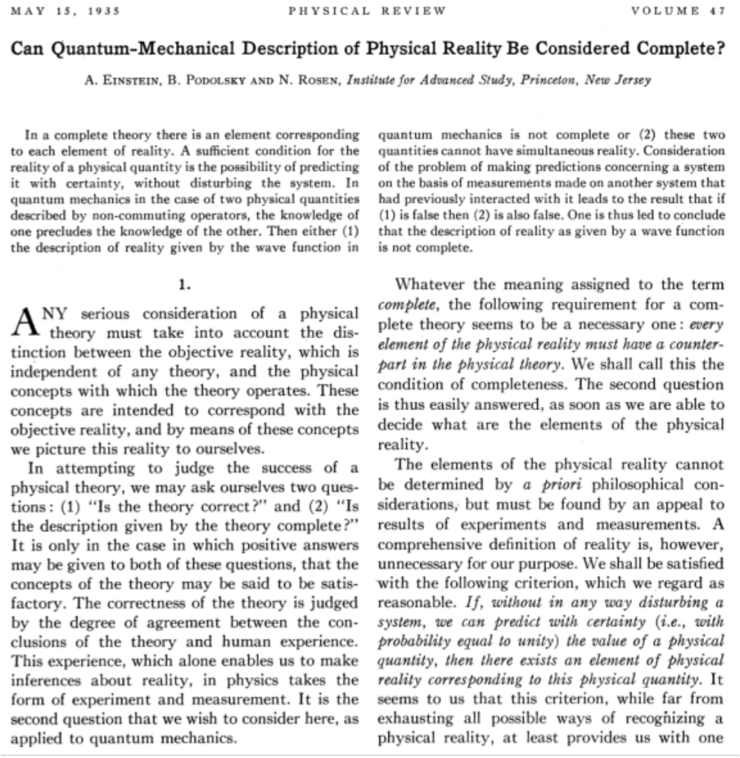
Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the

quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration

Einstein-Podolsky-Rosen (EPR) test



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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

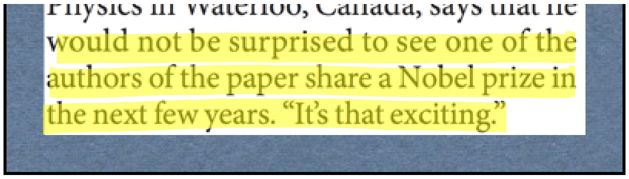
1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

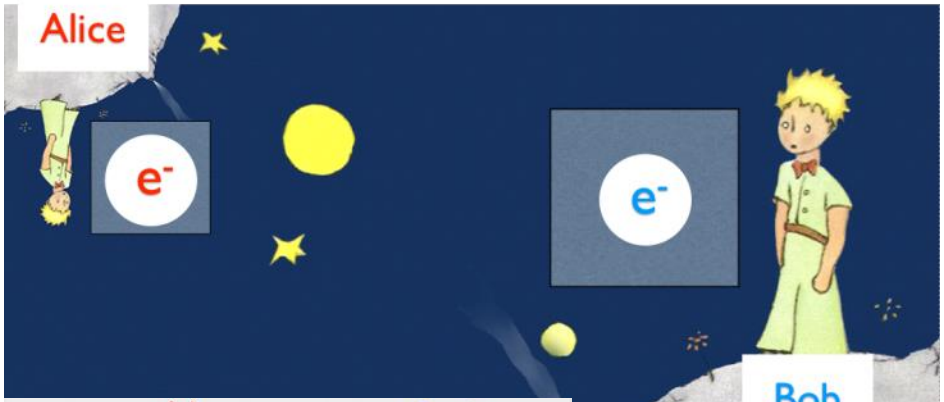
In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one



Physics in Waterloo, Canada, says that he would not be surprised to see one of the authors of the paper share a Nobel prize in the next few years. "It's that exciting."



Alice Bob

$$\left(\frac{1}{\sqrt{2}} |\uparrow\rangle \otimes |\downarrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\rangle \otimes |\uparrow\rangle \right)$$

Observe: If Alice measures ↑(up), then Bob measures ↓(down). And vice versa. Even if they are on different planets!

Is this magic?

No! It's not even interesting...

① It does not allow for faster-than-light comm.

② The same thing is possible classically.

Extension

Fact: If Alice and Bob measure in the same direction, then they always get opposite answers!

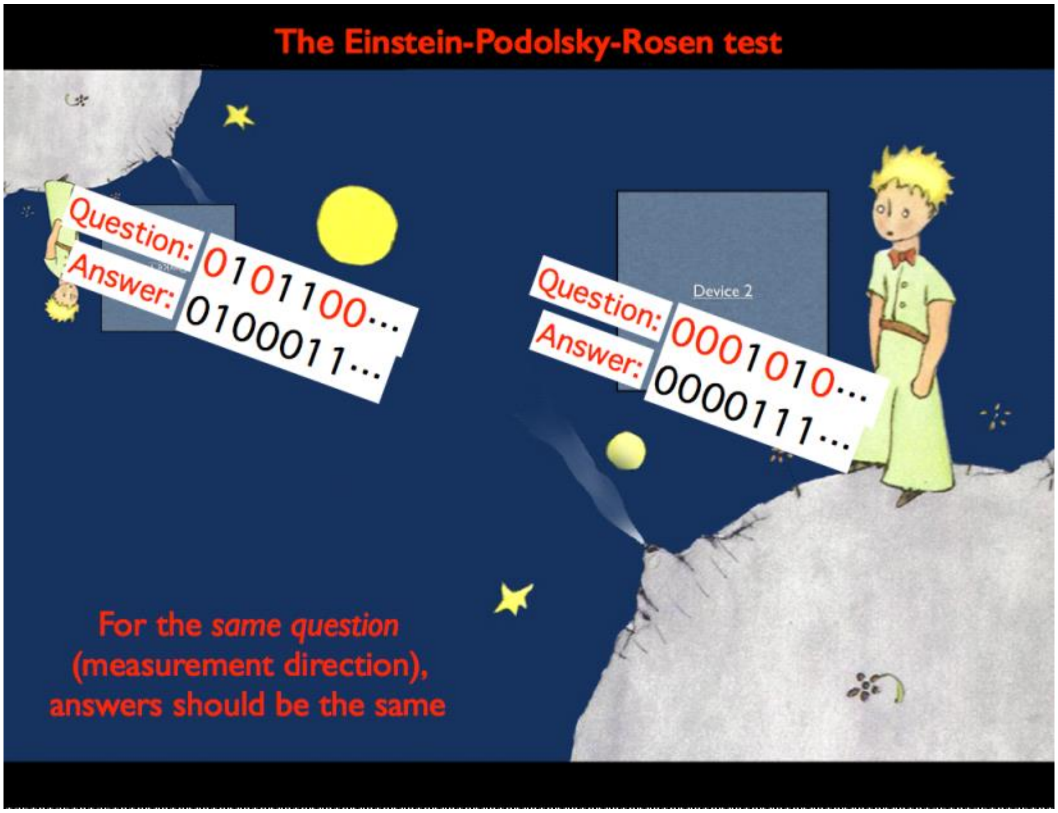
Is this spooky?

No!

All measurement outcomes could have been predetermined, when the electrons' state was created.

Bell inequality violations

Einstein-Podolsky-Rosen
Bell
CHSH
: When they measure in different directions, they'll see correlations that are impossible classically.



The Einstein-Podolsky-Rosen test

For the *same question* (measurement direction), answers should be the same

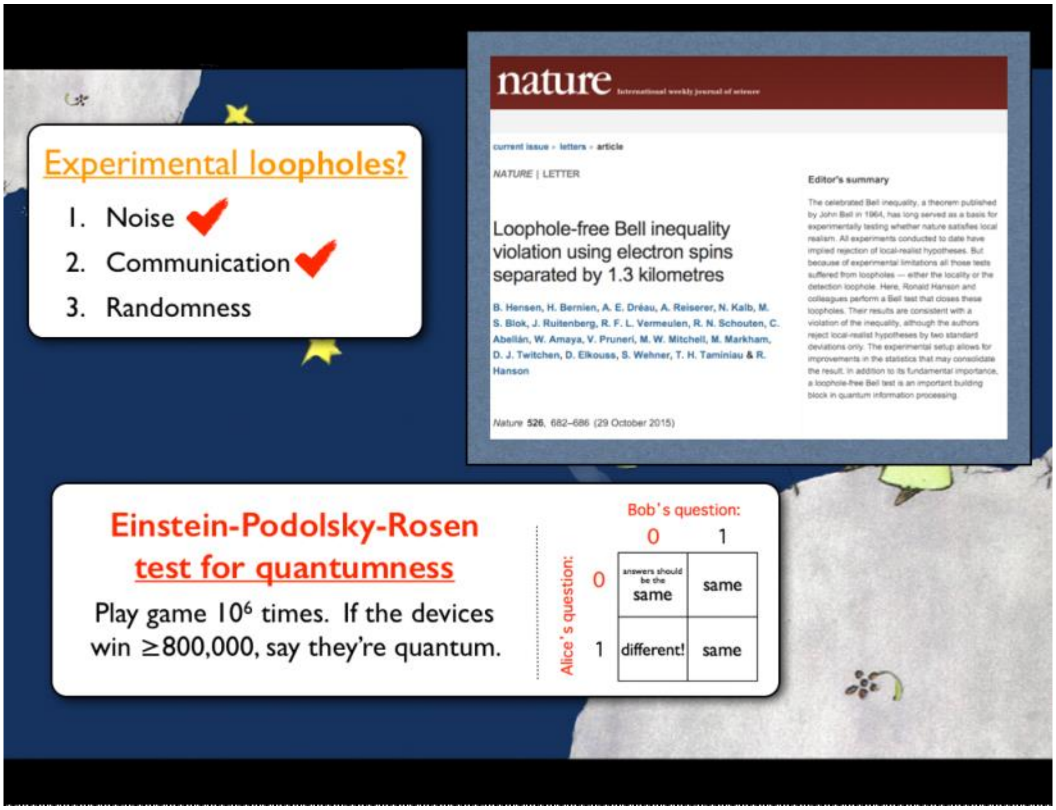
Bob's question:

Alice's question:

	0	1
0	answers should be the same	same
1	different!	same

Classical devices win with probability ≤ 75%

Entangled quantum devices can win with probability 85.4%



Experimental loopholes?

1. Noise ✓
2. Communication ✓
3. Randomness

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current issue » letters » article

NATURE | LETTER

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Einstein-Podolsky-Rosen test for quantumness

Play game 10^6 times. If the devices win ≥800,000, say they're quantum.

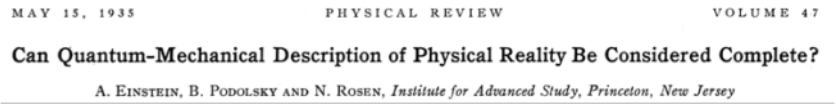
Bob's question:

Alice's question:

	0	1
0	answers should be the same	same
1	different!	same

What does a Bell inequality violation mean?

- Quantum entanglement is real
- There is no underlying classical physics explanation



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