Lecture 27: Spectral graph analysis

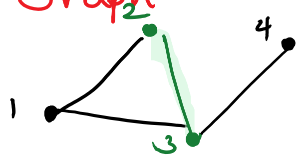
Tuesday, December 1, 2015 9:30 AM

Admin: Final exam is Thursday, Dec. 10, 11am–1pm
Course evaluations are online.

SPECTRAL ANALYSIS OF GRAPHS

Main idea: Study combinatorial objects — graphs — using algebraic tools — spectral decompositions of related matrices

Graph



Adjacency matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Laplacian

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

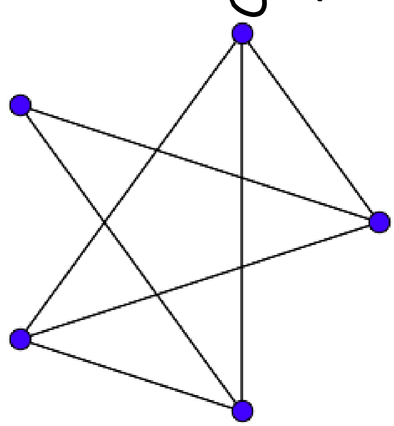
real, symmetric matrices
$\Rightarrow$ real e-values, real e-vectors

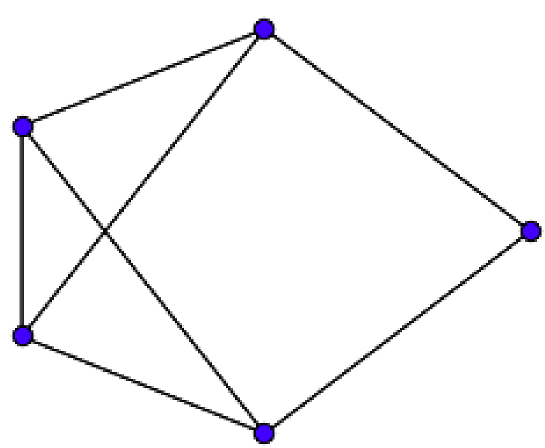
Random walk transition matrix

$$\left.\begin{pmatrix} 0 & 1/2 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 1 \\ 0 & 0 & 1/3 & 0 \end{pmatrix}\right\}\text{stochastic}$$

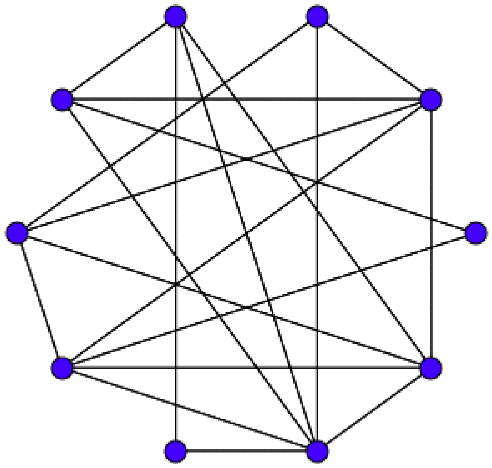
Example: GRAPH ISOMORPHISM

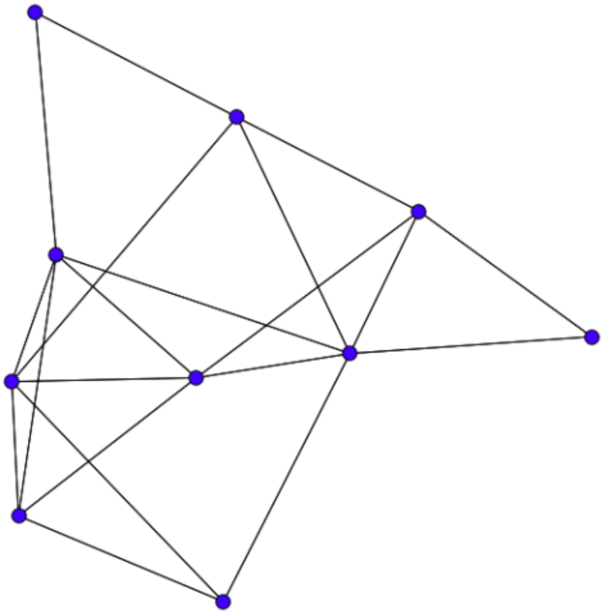
Are these graphs the same? (up to relabelling vertices)





What about these?





Combinatorial approach: Count vertices, the degrees of the vertices, ...
Algebraic approach:

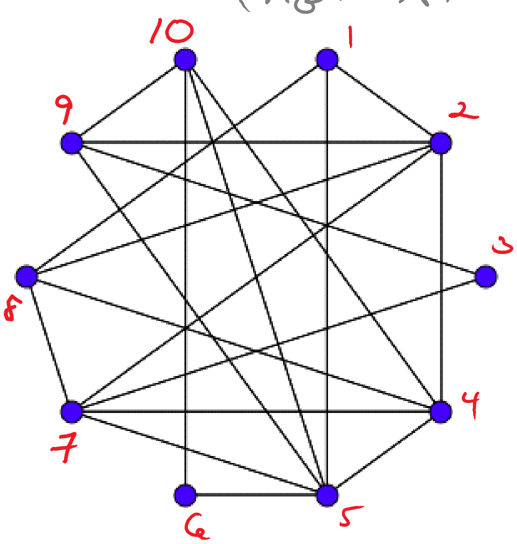
<u>**Observe**</u>: If graphs G and H are related by a vertex permutation, then

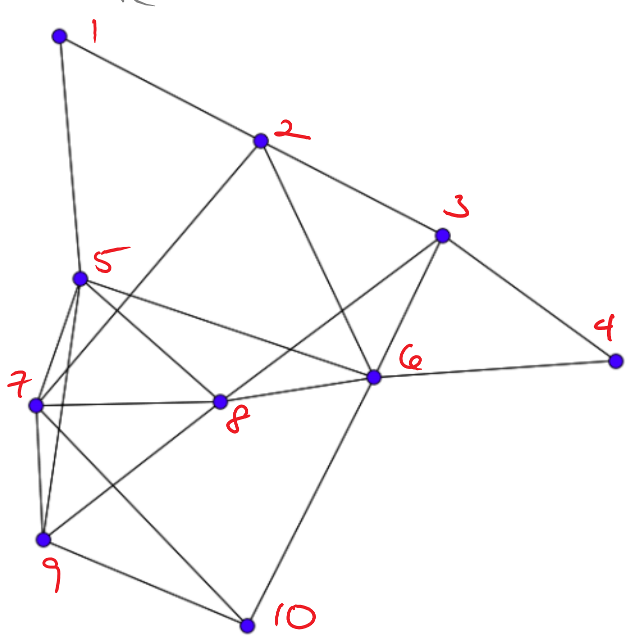
$$A_G = P A_H P^T$$

for a permutation matrix P

$\Rightarrow A_G$ and A_H have the same spectrum!

$$\left(A_G \vec{v} = \lambda \vec{v} \iff A_H(P^T\vec{v}) = \lambda(P^T\vec{v}) \right)$$





$$\text{Eigenvalues}\left[\begin{pmatrix} 0&1&0&0&1&0&0&1&0&0\\ 1&0&0&1&0&0&1&1&1&0\\ 0&0&0&0&0&0&1&0&1&0\\ 0&1&0&0&1&0&1&1&0&1\\ 1&0&0&1&0&1&1&0&1&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&1&1&1&1&0&0&1&0&0\\ 1&1&0&1&0&0&1&0&0&0\\ 0&1&1&0&1&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&1&0 \end{pmatrix} // N\right] \qquad \text{Eigenvalues}\left[\begin{pmatrix} 0&1&0&0&1&0&0&0&0&0\\ 1&0&1&0&0&1&1&0&0&0\\ 0&1&0&1&0&1&0&1&0&0\\ 0&0&1&0&0&1&0&0&0&0\\ 1&0&0&0&0&1&1&1&1&0\\ 0&1&1&1&1&0&0&1&0&1\\ 0&1&0&0&1&0&0&1&1&1\\ 0&0&1&0&1&1&1&0&1&0\\ 0&0&0&0&1&0&1&1&0&1\\ 0&0&0&0&0&1&1&0&1&0 \end{pmatrix} // N\right]$$

rixForm=

$$\begin{pmatrix} -3.03643\\ -1.79543\\ -1.46812\\ -0.932253\\ -0.587451\\ -2.87446\times 10^{-16}\\ 0.473082\\ 1.00677\\ 1.9756\\ 4.36423 \end{pmatrix}$$

ixForm=

$$\begin{pmatrix} -3.03643\\ -1.79543\\ -1.46812\\ -0.932253\\ -0.587451\\ 4.66604\times 10^{-16}\\ 0.473082\\ 1.00677\\ 1.9756\\ 4.36423 \end{pmatrix}$$

same spectrum ⇒ possibly isomorphic

Unfortunately, there exist non-isomorphic pairs of graphs with the same spectra.

No efficient algorithm is known for graph isomorphism.

Algorithms mix spectral with combinatorial techniques.

2013 IEEE 54th Annual Symposium on Foundations of Computer Science

Faster Canonical Forms For Strongly Regular Graphs (Extended Abstract)

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The University of Chicago

Combinatorics and Theoretical Computer Science seminar

Date: **Tuesday, November 10, 2015**
Time: 3:00pm
Place: Kent 120

Speaker: László Babai (The University of Chicago)

Title: **Graph Isomorphism in Quasipolynomial Time I: The "Local Certificates algorithm"**

Abstract:

In a series of two talks we outline an algorithm that solves the Graph Isomorphism (GI) problem and the related problems of String Isomorphism (SI) and Coset Intersection (CI) in quasipolynomial ($\exp(\mathrm{polylog}\, n)$) time.

The best previous bound for GI was $\exp(\sqrt{n \log n})$, where n is the number of vertices (Luks, 1983). For SI and CI the best previous bound was similar, $\exp(\sqrt{n}(\log n)^c)$, where n is the size of the permutation domain (the speaker, 1983).

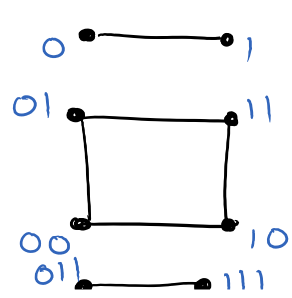
In this first talk we give an overview of the algorithm and present the core group-theoretic divide-and-conquer routine, the "Local Certificates algorithm." Familiarity with undergraduate-level group theory will be assumed.

Two Examples: Hypercube and Cycle

Example: The Hypercube

n = 1 :

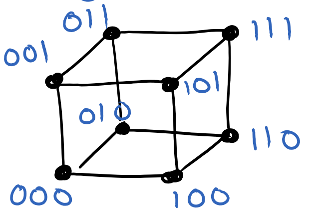
n = 2 :



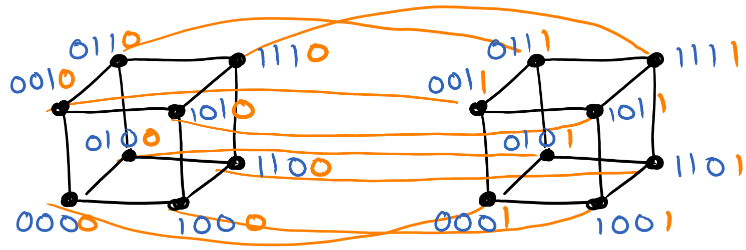
vertices → bit strings of length n

$x \underset{\text{adjacent to}}{\sim} y$ ⟷ they differ in one coordinate

$n=3$:



$n=4$:



Adjacency matrix:

$$A = \sum_{\text{edges}(u,v)} \left(\vec{e}_u \vec{e}_v^{\,T} + \vec{e}_v \vec{e}_u^{\,T} \right)$$

$$A_{H_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A_{H_2} = \begin{array}{c} \\ 00 \\ 10 \\ 01 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$A_{H_3} = \begin{array}{c} \\ 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 000 & 100 & 010 & 110 & 001 & 101 & 011 & 111 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & & & \\ 1 & 0 & 0 & 1 & & 1 & & \\ 1 & 0 & 0 & 1 & & & 1 & \\ 0 & 1 & 1 & 0 & & & & 1 \\ 1 & & & & 0 & 1 & 1 & 0 \\ & 1 & & & 1 & 0 & 0 & 1 \\ & & 1 & & 1 & 0 & 0 & 1 \\ & & & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Recall: Tensor products

Vectors:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 \cdot q_1 \\ p_1 \cdot q_2 \\ p_2 \cdot q_1 \\ p_2 \cdot q_2 \end{pmatrix}$$

$$\boxed{(\vec{u} \otimes \vec{v})_{(i,j)} = u_i v_j}$$

Operators:

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots \\ a_{21} B & a_{22} B & \\ \vdots & & \ddots \end{pmatrix}$$

$$\boxed{(A \otimes B)_{(i,j),(k,l)} = A_{ik} B_{jl}}$$

$$\Rightarrow A_{H_n} = I_2 \otimes A_{H_{n-1}} + A_{H_1} \otimes I_{2^{n-1}}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2^{\otimes(n-1)} + I_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2^{\otimes(n-2)}$$

$$+ \cdots + I_2^{\otimes(n-1)} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem: What are the eigenvalues of A_{H_n}?

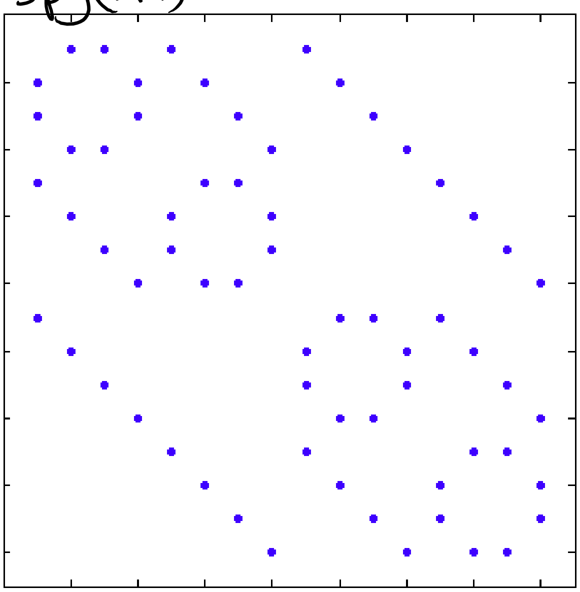
Step one: experiment!

```
>> X = [0 1; 1 0];
I = eye(2);
>> kron(X,I)

ans =

     0     0     1     0
     0     0     0     1
     1     0     0     0
     0     1     0     0

>> kron(I,X)

ans =

     0     1     0     0
     1     0     0     0
     0     0     0     1
     0     0     1     0

>> X2 = kron(X,I) + kron(I,X)

X2 =

     0     1     1     0
     1     0     0     1
     1     0     0     1
     0     1     1     0
```

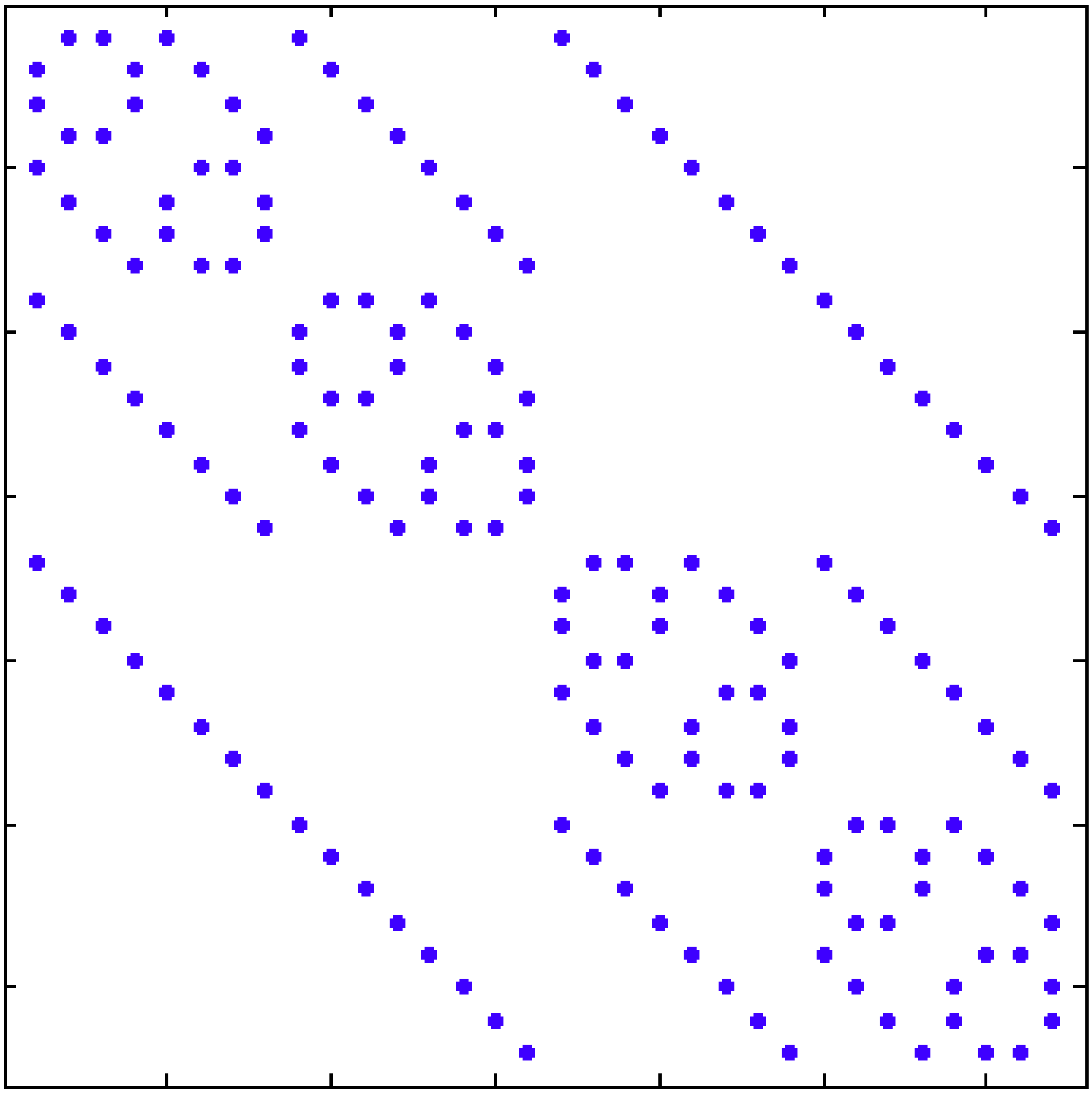
```
>> X3 = kron(X,eye(2^2)) + kron(I,X2)

X3 =

     0     1     1     0     1     0     0     0
     1     0     0     1     0     1     0     0
     1     0     0     1     0     0     1     0
     0     1     1     0     0     0     0     1
     1     0     0     0     0     1     1     0
     0     1     0     0     1     0     0     1
     0     0     1     0     1     0     0     1
     0     0     0     1     0     1     1     0
```

```
>> X4 = kron(X,eye(2^3)) + kron(I,X3)

X4 =

     0     1     1     0     1     0     0     0     1     0     0     0     0     0     0     0
     1     0     0     1     0     1     0     0     0     1     0     0     0     0     0     0
     1     0     0     1     0     0     1     0     0     0     1     0     0     0     0     0
     0     1     1     0     0     0     0     1     0     0     0     1     0     0     0     0
     1     0     0     0     0     1     1     0     0     0     0     0     1     0     0     0
     0     1     0     0     1     0     0     1     0     0     0     0     0     1     0     0
     0     0     1     0     1     0     0     1     0     0     0     0     0     0     1     0
     0     0     0     1     0     1     1     0     0     0     0     0     0     0     0     1
     1     0     0     0     0     0     0     0     0     1     1     0     1     0     0     0
     0     1     0     0     0     0     0     0     1     0     0     1     0     1     0     0
     0     0     1     0     0     0     0     0     1     0     0     1     0     0     1     0
     0     0     0     1     0     0     0     0     0     1     1     0     0     0     0     1
     0     0     0     0     1     0     0     0     1     0     0     0     0     1     1     0
     0     0     0     0     0     1     0     0     0     1     0     0     1     0     0     1
     0     0     0     0     0     0     1     0     0     0     1     0     1     0     0     1
     0     0     0     0     0     0     0     1     0     0     0     1     0     1     1     0
```

spy(X4):



spy(X5):



Step two: Solve for the eigenvalues!

Proposition: If A's eigenvalues are $\lambda_1, \dots, \lambda_m$
and B's eigenvalues are $\sigma_1, \dots, \sigma_n$,
then the eigenvalues of $A \otimes I_n + I_m \otimes B$ are

$$\lambda_1+\sigma_1, \lambda_1+\sigma_2, \dots, \lambda_1+\sigma_n,$$
$$\lambda_2+\sigma_1, \dots, \lambda_2+\sigma_n,$$
$$\dots,$$
$$\lambda_m+\sigma_1, \dots, \lambda_m+\sigma_n$$

i.e. all sums (eigenvalue of A) + (eigenvalue of B).

Proof:

① Eigenvalues of $A \otimes I_n$:

$$\underbrace{\lambda_1, \lambda_1, \dots, \lambda_1}_{n \text{ times}}, \quad \dots, \quad \underbrace{\lambda_m, \dots, \lambda_m}_{n \text{ times}}$$

since I_n has e-value 1 with multiplicity n.

② Eigenvalues of $I_m \otimes B$:

σ_1 (m times), ..., σ_n (m times)

③ $A \otimes I_n$ and $I_m \otimes B$ commute!

$$\begin{aligned}(A \otimes I)(I \otimes B) &= (AI) \otimes (IB) \\ &= A \otimes B \\ &= (I \otimes B)(A \otimes I)\end{aligned}$$

$\Rightarrow$ they can be simultaneously diagonalized.

If $A\vec{u}_i = \lambda_i \vec{u}_i$

$B\vec{v}_j = \sigma_j \vec{v}_j$,

then $(A \otimes I + I \otimes B)(\vec{u}_i \otimes \vec{v}_j) = (\lambda_i + \sigma_j)(\vec{u}_i \otimes \vec{v}_j)$ ✓ $\square$

Theorem: The eigenvalues of the adjacency matrix for the n-dimensional hypercube H_n are:

n

$n-2$ w/multiplicity $\binom{n}{1} = n$

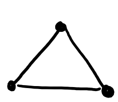
$n-4$ w/multiplicity $\binom{n}{2}$

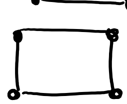
$\vdots$

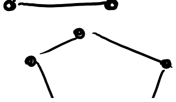
$-n+2$

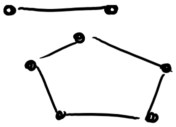
$-n$

Example: THE CYCLE

C_3  $A_{C_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

C_4  $A_{C_4} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

C_5  $A_{C_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ \dots \end{pmatrix}$

C_5 

$$A_{C_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$A_{C_n} = T_n + T_n^T$, where $T_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}$

$= e_2 e_1^T + e_3 e_2^T + \cdots + e_n e_{n-1}^T + e_1 e_n^T$

the cyclic shift/translation, an orthogonal (permutation) matrix

Observe: "Waves are eigenvectors of translations"

Precisely: Let $\omega_n = e^{2\pi i/n}$

$$\vec{u}_k = \tfrac{1}{\sqrt{n}}(1, \omega^k, \omega^{2k}, \omega^{3k}, \dots, \omega^{(n-1)k})$$

$$\Rightarrow T\vec{u}_k = \tfrac{1}{\sqrt{n}}(\omega^{(n-1)k}, 1, \omega^k, \omega^{2k}, \dots, \omega^{(n-2)k})$$

$$= \underbrace{\omega^{-k}}_{\text{an eigenvector!}} \cdot \vec{u}_k$$

(e.g., $k=0$: $T\frac{1}{\sqrt{n}}(1,1,\dots,1) = \frac{1}{\sqrt{n}}(1,1,\dots,1)$.)

Corollary: The wave vectors $\vec{u}_0, \vec{u}_1, \dots, \vec{u}_{n-1}$ form an orthonormal basis.

(This is called the Fourier basis.)

Proof: It is simple enough to calculate the inner products

$$\vec{u}_j \cdot \vec{u}_k = u_j^\dagger u_k \underset{\text{(we claim)}}{=} \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}.$$

But, even easier: They are e-vectors of a normal matrix (T) with distinct e-values. $\Rightarrow$ they must be orthogonal.

The normalization $\frac{1}{\sqrt{n}}$ is obvious. ✓ $\square$

Corollary: The matrix

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}$$

is unitary.

Diagonalizing the cycle:

$$A_{C_n} = T_n + T_n^{-1}$$

- T_n and T_n^{-1} both commute with T_n
  (any matrix commutes with powers of itself)

$\Rightarrow [A_{C_n}, T_n] = 0$

$\Rightarrow$ They can be simultaneously diagonalized

$\Rightarrow$ (since the eigenspaces of T_n are all 1D)
the waves $\vec{u}_k$ are the eigenvectors of A_{C_n}

$$A_{C_n}\vec{u}_k = T_n \vec{u}_k + T_n^{-1}\vec{u}_k$$
$$= \left(\omega_n^{-k} + \omega_n^{k}\right)\vec{u}_k$$
$$\underbrace{\omega_n^{-k} + \omega_n^{k}}_{= e^{-2\pi i k/n} + e^{2\pi i k/n}} = \boxed{2\cos\frac{2\pi k}{n}}$$

largest e-value: 2 (for $k=0$)

2^{nd} largest e-value: $2\cos\frac{2\pi(n-1)}{n} \approx 2\left(1-\left(\frac{2\pi}{n}\right)^2\right)$

$\Rightarrow$ spectral gap $= \Theta\left(\frac{1}{n^2}\right)$

smallest e-value: (for $k = \frac{n}{2}$) $\quad -2$ if n even

$2\cos\left(\pi\left(1-\frac{1}{n}\right)\right)$ if n odd
$\approx -2 + \frac{\pi^2}{n^2}$

Observe: The same argument holds for any "circulant matrix,"

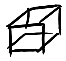
a matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{n-2} & a_{n-1} & a_0 & a_1 & a_2 & \cdots \\ \vdots & & & & & \\ a_1 & a_2 & a_3 & \cdots & & a_0 \end{pmatrix}$$

Since $A = a_0 I + a_1 T_n^{-1} + a_2 T_n^{-2} + \cdots + a_{n-1} T_n$,
it commutes with T_n $\Rightarrow$ It is diagonal in the Fourier basis.
(Of course this can also be verified directly.)

COMPARISON:

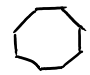
COMPARISON:

$\frac{1}{n} A_{H_n}$ 

$1, 1-\frac{2}{n}, 1-\frac{4}{n}, \dots, -1$

spectral gap $\lambda_1 - \lambda_2$

$\frac{2}{n} = \frac{2}{\log_2(\#\text{ of vertices})}$

$\frac{1}{2} A_{C_n}$ 

$1, \cos\frac{2\pi}{n} \approx 1 - \frac{2\pi^2}{n^2}, \cos\frac{4\pi}{n}, \dots$

spectral gap

$\approx \frac{2\pi^2}{(\#\text{ of vertices})^2}$

Moral: Better-connected graph
$\Rightarrow$ Larger spectral gap

($\Rightarrow$ faster convergence for random walks, Page Rank, power method)

Theorem: "Cheeger's inequality"

Let G be a graph where every vertex has degree d.
List the eigenvalues of $\frac{1}{d} A_G$:

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

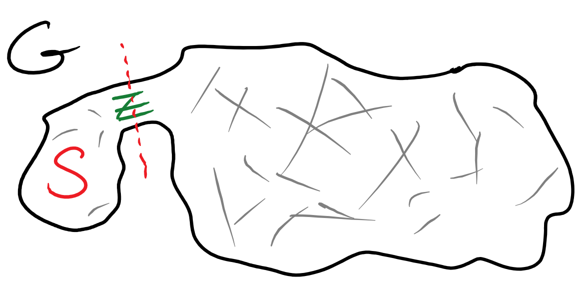
Then

$$\frac{1-\lambda_2}{2} \leq \frac{1}{d}\left(\min_{\substack{S \text{ a subset} \\ \text{of at most } \frac{1}{2} \\ \text{the vertices}}} \frac{\#\text{ of edges leaving } S}{|S|} \right) \leq \sqrt{2(1-\lambda_2)}$$

Thus the spectral gap $1-\lambda_2$ is small

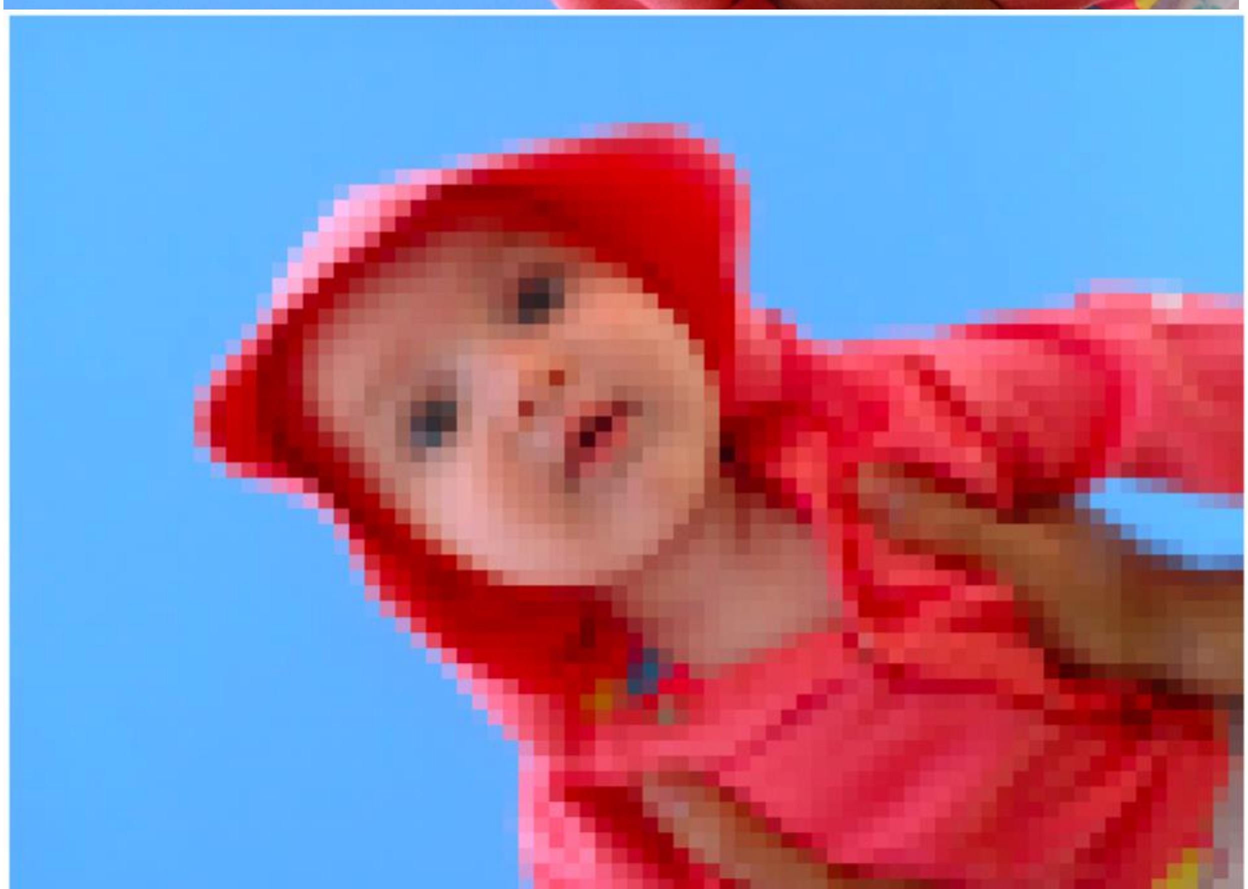
$\Updownarrow$

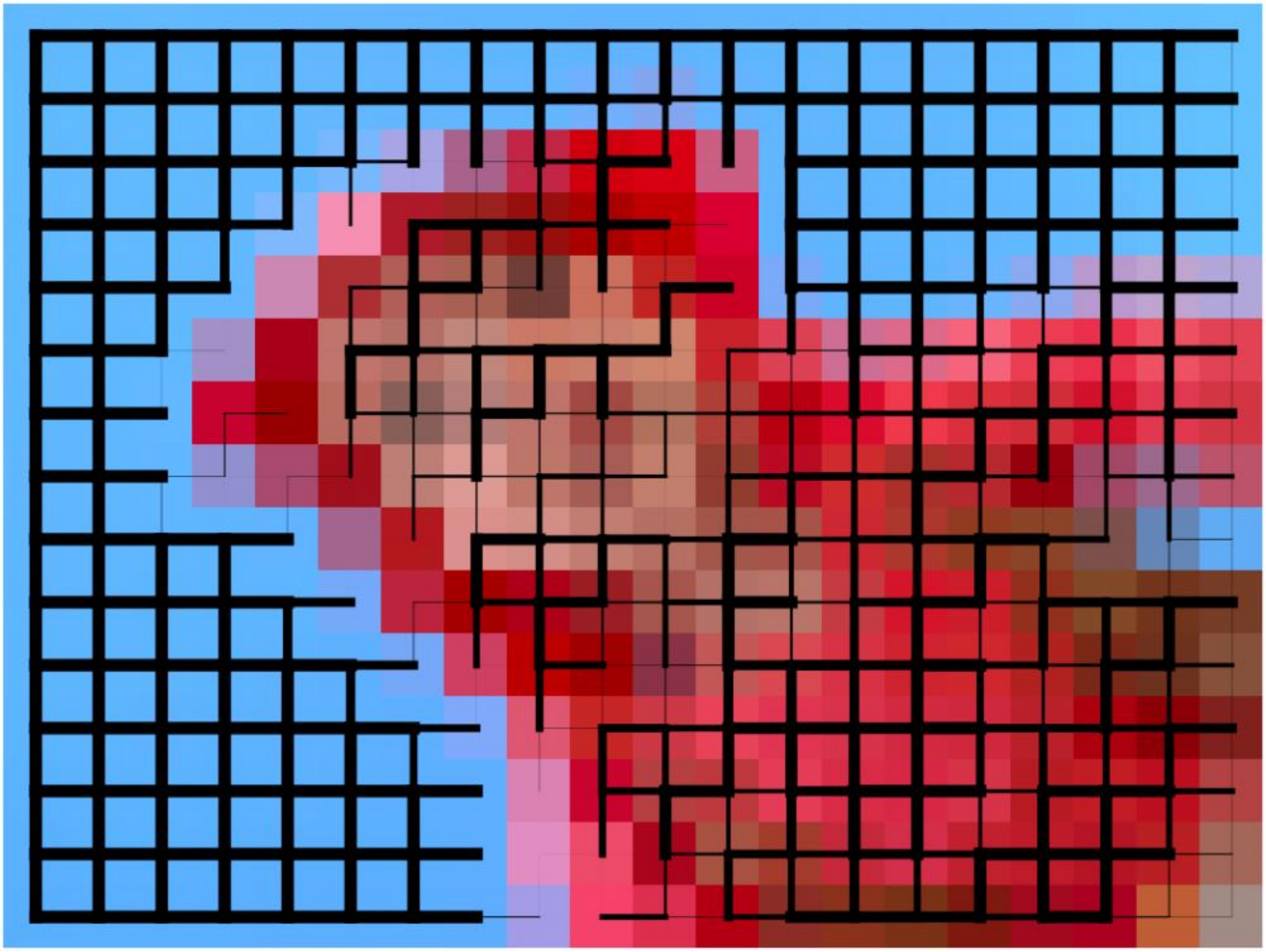
there is a cut in G crossed by relatively few edges



We won't prove Cheeger's Inequality, but let's consider some conceptual applications.

Application 1: Spectral Image Segmentation





APPLICATIONS 2,3: SPECTRAL GRAPH DRAWING
SPECTRAL CLUSTERING

Spectral graph layout

Idea: To embed a graph on a line, solve

minimize $\sum_{\text{edges}(i,j)} \|\vec{x}(i) - x(j)\|^2$

subject to $\sum_i \|\vec{x}(i)\|^2 = 1$ ⟵ to prevent $\vec{x}(i) \equiv 0$

$\sum_i \vec{x}(i) = 0$ ⟵ to prevent $\vec{x}(i) \equiv \text{constant}$

But the Laplacian gives exactly this quadratic form

$$\vec{x}^T L \vec{x} = \sum_{\text{edges}(i,j)} (x_i - x_j)^2$$

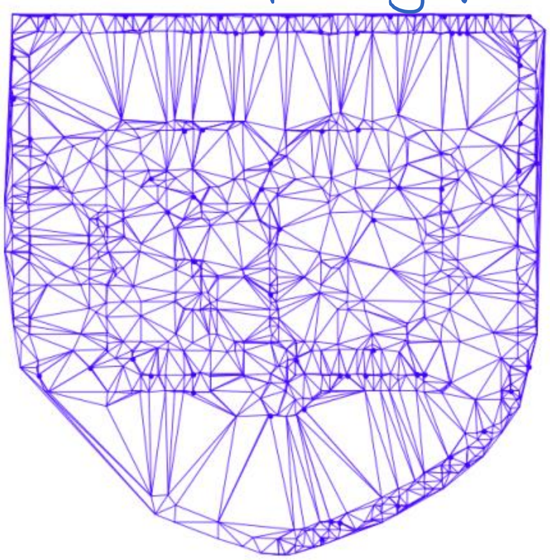
while $\sum_i x_i = 0$ just means $\vec{x}$ is perpendicular to the all-ones vector (the e-value 0 e-vector of L).

⇒ Optimal embedding uses L's eigenvector with the second smallest eigenvalue.

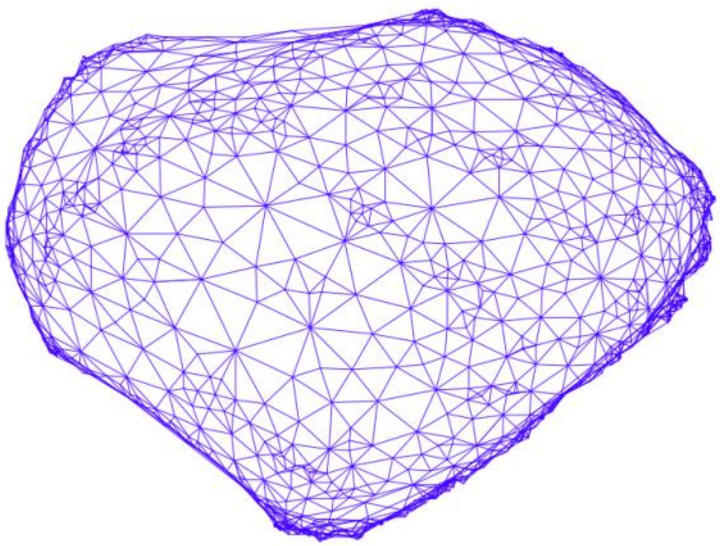
Can use more eigenvectors for higher-dimensional embeddings.

Example:

A certain planar graph:



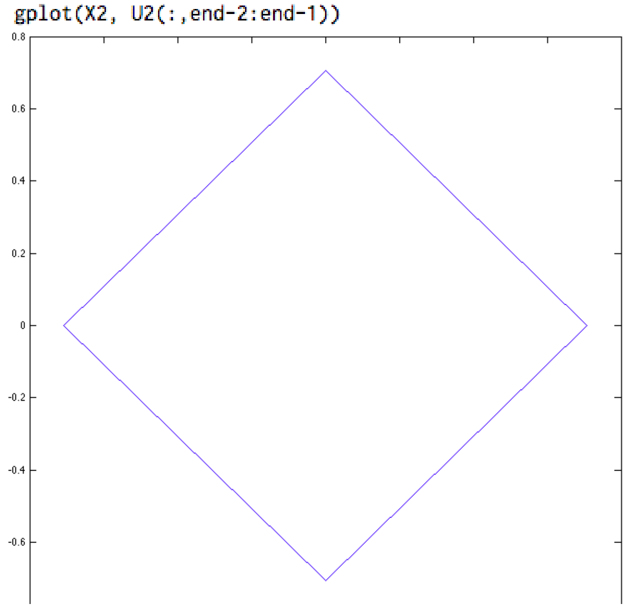
A spectral embedding: plot vertex i at $(v_2(i), v_3(i))$.



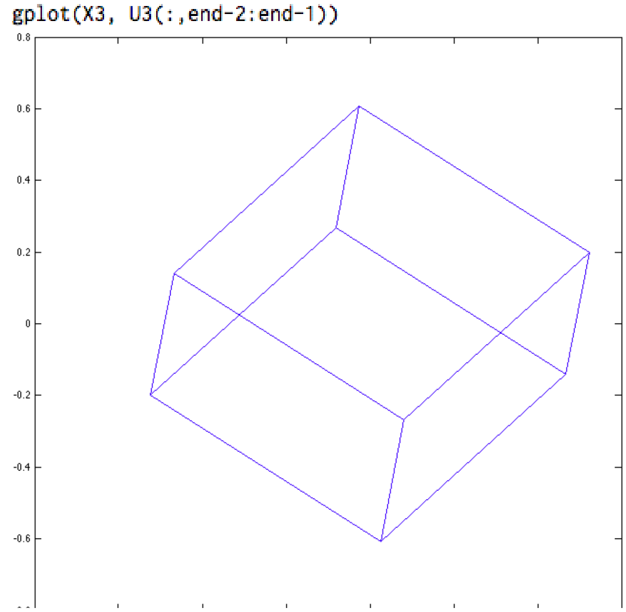
Example:

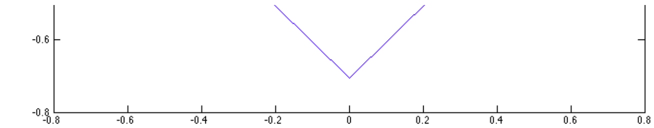
```
X = [0 1; 1 0];
I = eye(2);
X2 = kron(X,I) + kron(I,X);
X3 = kron(X,eye(2^2)) + kron(I,X2);
X4 = kron(X,eye(2^3)) + kron(I,X3);
[U2,D2] = eig(X2);
[U3,D3] = eig(X3);
[U4,D4] = eig(X4);
```

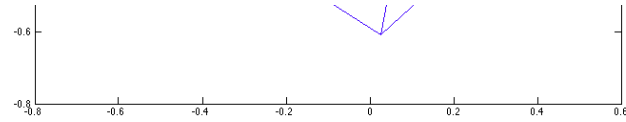
```
gplot(X2, U2(:,end-2:end-1))
```



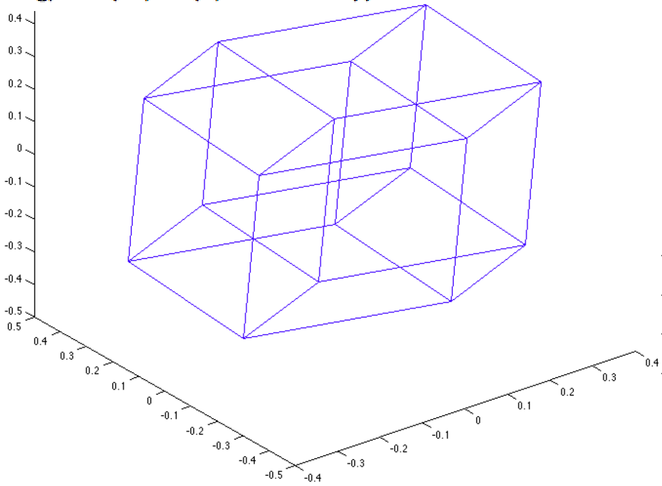

```
gplot(X3, U3(:,end-2:end-1))
```





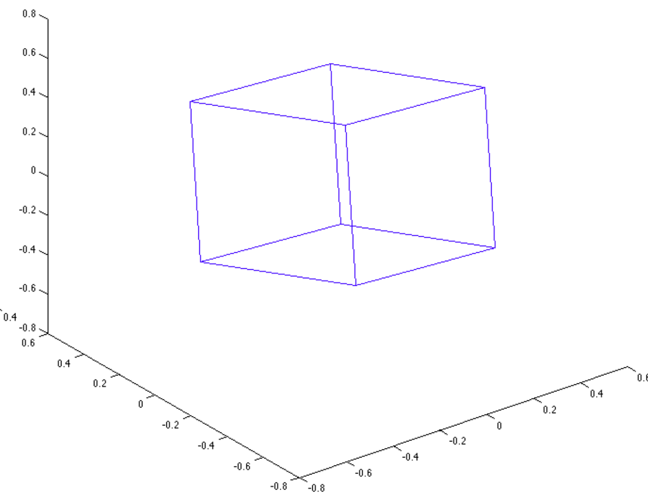


gplot3(X4, U4(:,end-3:end-1))



gplot3(X3, U3(:,end-3:end-1))

3 e-vectors



Spectral clustering

Problem: Given a graph, divide its vertices into k sets that are "good clusters".

Standard approach:

1) Using eigenvectors 2 to $d+1$ of the adjacency/Laplacian matrix, embed the vertices into $\mathbb{R}^d$.
2) Then ignore the edges and use a metric clustering algorithm (eg., k-means) to cluster the points in $\mathbb{R}^d$.

This works really well!!!

But except for dividing into $k=2$ clusters, nobody really knows why.

arXiv:1301.5584v1 [cs.DS] 23 Jan 2013

Improved Cheeger's Inequality: Analysis of Spectral Partitioning Algorithms through Higher Order Spectral Gap

Tsz Chiu Kwok* Lap Chi Lau† Yin Tat Lee‡ Shayan Oveis Gharan§
Luca Trevisan¶

Abstract

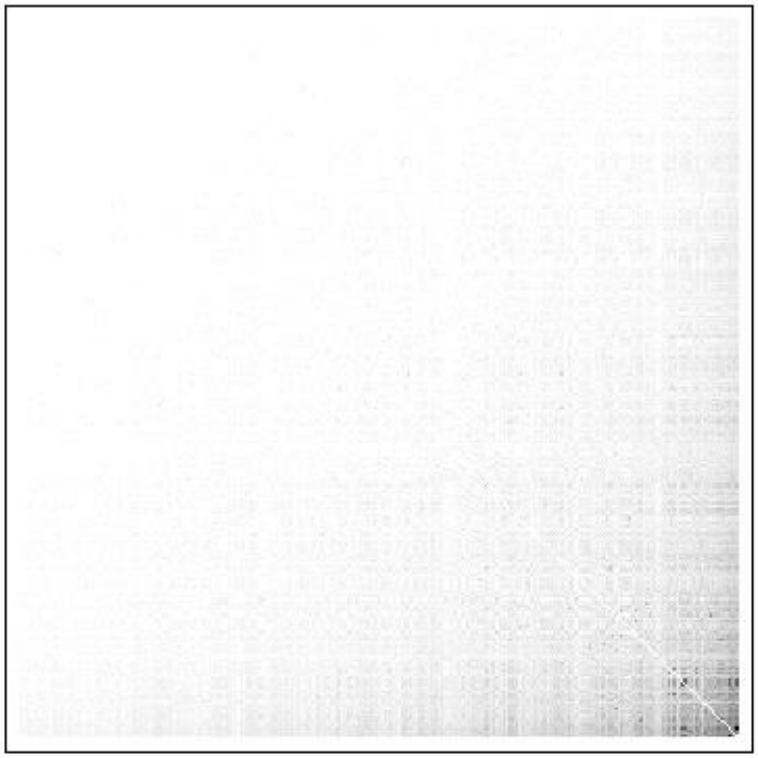
Let $\phi(G)$ be the minimum conductance of an undirected graph G, and let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized Laplacian matrix of G. We prove that for any graph G and any $k \geq 2$,

$$\phi(G) = O(k)\frac{\lambda_2}{\sqrt{\lambda_k}},$$

and this performance guarantee is achieved by the spectral partitioning algorithm. This improves Cheeger's inequality, and the bound is optimal up to a constant factor for any k. Our result shows that the spectral partitioning algorithm is a constant factor approximation algorithm for finding a sparse cut if λ_k is a constant for some constant k. This provides some theoretical justification to its empirical performance in image segmentation and clustering problems. We extend the analysis to other graph partitioning problems, including multi-way partition, balanced separator, and maximum cut.

Example: This is how I processed the Netflix movies dataset in lecture 1:

```
matrix = Import["/Users/breic/Desktop/adjacencymatrix.txt", "Table"];
matrix += Transpose[matrix];
matrix // ArrayPlot
```



```
laplacian = DiagonalMatrix[Plus @@ # & /@ matrix] - matrix // N;
di = DiagonalMatrix[(1 / Plus @@ #) & /@ matrix // N];
evs = Eigensystem[√di .laplacian.√di] // Transpose // Reverse;

coordinates = evs[[2 ;; 9, 2]] // Transpose;

numclusters = 16;

ClusteringComponents[coordinates, numclusters, 1, Method → "PAM"]
```

embedding into $\mathbb{R}^d$

Indiana Jones and the L
Lord of the Rings: The
Lord of the Rings: The
Lord of the Rings: The
Raiders of the Lost Ark ,
Star Wars: Episode IV:
Star Wars: Episode VI:
Star Wars: Episode V: T
The Lord of the Rings:

A Walk to Remember
Coyote Ugly
Dirty Dancing
How to Lose a Guy in 10
Maid in Manhattan
Pretty Woman ,
Sister Act
The Princess Diaries 2:
The Princess Diaries (W
The Wedding Planner
What Women Want

Bend It Like Beckham
Bridget Jones's Diary
Frida
Life Is Beautiful
Love Actually
Moulin Rouge ,
My Big Fat Greek Weddin
Pride and Prejudice
Rabbit-Proof Fence
Shakespeare in Love
Whale Rider

Con Air
Double Jeopardy
Gone in 60 Seconds
Independence Day
Lethal Weapon 4
Men in Black II ,
Pearl Harbor
The Fast and the Furiou
The Patriot
Tomb Raider
Twister

12 Angry Men
Airplane!
American Pie
American Pie 2
Austin Powers in Goldme
Austin Powers: Internat
Austin Powers: The Spy ,
Interview with the Vamp
Liar Liar
Meet the Parents
Ransom
Spaceballs
Spider-Man
Wayne's World

A Bug's Life
Breakfast at Tiffany's
City of Angels
Ever After: A Cinderell
Finding Nemo (Widescree
Grease
Harry Potter and the Ch
Harry Potter and the Pr ,
Harry Potter and the So
Runaway Bride
The Lion King: Special
The NeverEnding Story
The Princess Bride
The Sound of Music
Willy Wonka & the Choco

Amelie
American Beauty
Being John Malkovich
Crouching Tiger
Election
Eternal Sunshine of the
High Fidelity
Lock ,
Lost in Translation
Magnolia
Run Lola Run
Rushmore
Sideways
The Royal Tenenbaums
Y Tu Mama Tambien

2001: A Space Odyssey
All the President's Men
Blade Runner
Gandhi
Jaws
L.A. Confidential
Lawrence of Arabia
Lord of the Rings: The
One Flew Over the Cucko ,
Seven Samurai
The Aviator
The Exorcist
The Godfather
The Godfather
The Graduate
The Great Escape
The Maltese Falcon

Cold Mountain
Collateral
Crash
Fahrenheit 9/11
Finding Neverland
Hotel Rwanda
Man on Fire
Master and Commander: T
Million Dollar Baby
Ocean's Twelve
Ray
Road to Perdition
Runaway Jury
Seabiscuit
The Manchurian Candidat
The Notebook
The Phantom of the Oper
The Pianist

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A Fish Called Wanda
Alien: Collector's Edit
Back to the Future
Back to the Future Part
Batman
Die Hard 2: Die Harder
Die Hard With a Vengean
Goldfinger
Groundhog Day
Indiana Jones and the T
Men in Black
Mission: Impossible
Predator: Collector's E
Rocky
Speed
Star Trek II: The Wrath
Terminator 2: Extreme E
The Hunt for Red Octobe
The Terminator
True Lies

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A Knight's Tale
Ice Age
Jurassic Park
Lara Croft: Tomb Raider
Minority Report
Pirates of the Caribbea
Rush Hour
Rush Hour 2
Sleeping Beauty: Specia
Spider-Man 2
Star Wars: Episode II:
Star Wars: Episode I: T
Terminator 3: Rise of t
The Fifth Element
The Incredibles
The Matrix
The Matrix: Reloaded
The Matrix: Revolutions
The Mummy
The Mummy Returns
X2: X-Men United
X-Men

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Adaptation
A Few Good Men
Air Force One
Armageddon
Clear and Present Dange
Crimson Tide
Enemy of the State
Entrapment
High Crimes
In the Line of Fire
Lethal Weapon
Lethal Weapon 2
Lethal Weapon 3
Patriot Games
Rules of Engagement
Swordfish
The Bone Collector
The Client
The Fugitive
The Negotiator
The Pelican Brief
The Rock
The Sum of All Fears

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12 Monkeys
Almost Famous
American History X
Anchorman: The Legend o
Donnie Darko
Garden State
GoodFellas: Special Edi
Grosse Pointe Blank
Heat: Special Edition
Kill Bill: Vol. 1
Kill Bill: Vol. 2
Memento
Napoleon Dynamite
Office Space
Pulp Fiction
Requiem for a Dream
Reservoir Dogs
Seven
Sin City

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Ace Ventura: Pet Detect
A League of Their Own
A River Runs Through It
Basic Instinct
Cheaper by the Dozen
Daddy Day Care
Erin Brockovich
Face/Off
Father of the Bride
Kindergarten Cop
Legally Blonde
Mrs. Doubtfire
Notting Hill
Pay It Forward
Phenomenon
Serendipity
Shall We Dance?
Sleepless in Seattle
Steel Magnolias

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50 First Dates
Anger Management
Bad Boys II
Behind Enemy Lines
Bruce Almighty
Dodgeball: A True Under
Harold and Kumar Go to
Hero
Hidalgo
Hitch
Hostage
I Robot
Meet the Fockers
National Treasure
Ocean's Eleven
Sahara
Shrek 2
The Bourne Identity
The Bourne Supremacy
The Count of Monte Cris

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Apollo 13
As Good as It Gets
Black Hawk Down
Boys Don't Cry
Cast Away
Chocolat
Dances With Wolves: Spe
Dead Man Walking
Driving Miss Daisy
Enemy at the Gates
E.T. the Extra-Terrestr
Field of Dreams
Forrest Gump
Fried Green Tomatoes
Gladiator
Glory
Good Will Hunting
Jerry Maguire
Moonstruck
My Cousin Vinny
October Sky
Philadelphia
Primal Fear
Rain Man
Remember the Titans

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