Lecture 28: Review

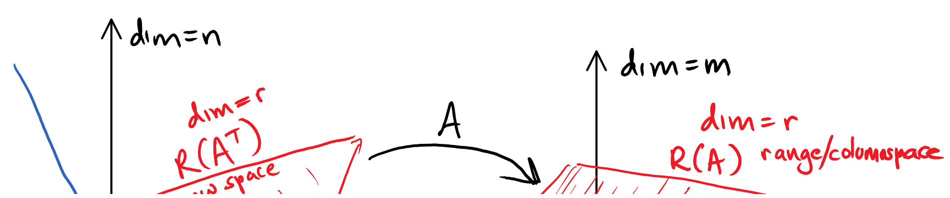
Thursday, December 3, 2015 9:30 AM

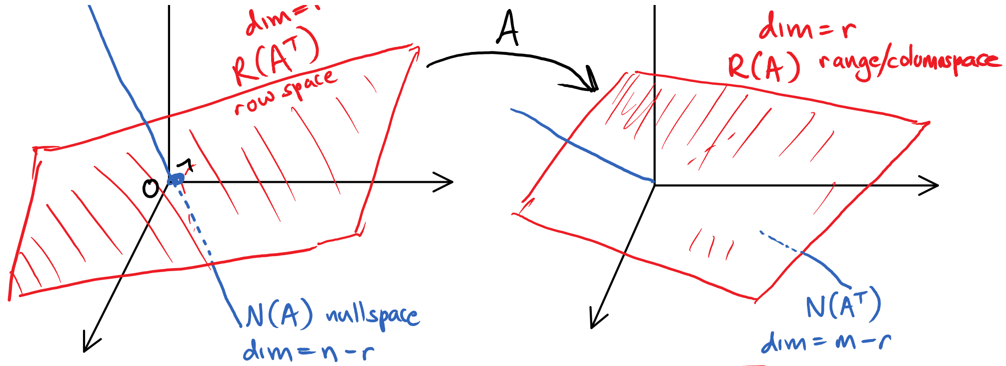
Admin: Final exam is Thursday, Dec. 10, 11am–1pm
You are allowed to use two pages of notes.
Course evaluations are online.

Topics since the midterm

- Singular-value decomposition
- Spectral decomposition
- Least squares: $\min_x \|Ax-b\|$
  - either using normal equations or the pseudoinverse
- Special matrices
  - Diagonalizable
  - Normal
  - Unitary, orthogonal (and isometries)
  - Symmetric, Hermitian
  - Positive semi-definite, positive definite
  - Stochastic
- Condition number, stability of linear systems of equations
- Power method for finding eigenvectors
  Spectral gap
- Applications:
  - Solving recurrences and differential equations
  - Linear regression
  - Principal component analysis ← low-rank approx. of matrices
  - PageRank ← Markov chains
  - Quantum physics ← unitary & Hermitian matrices, tensor products, commuting matrices
  - Spectral graph analysis

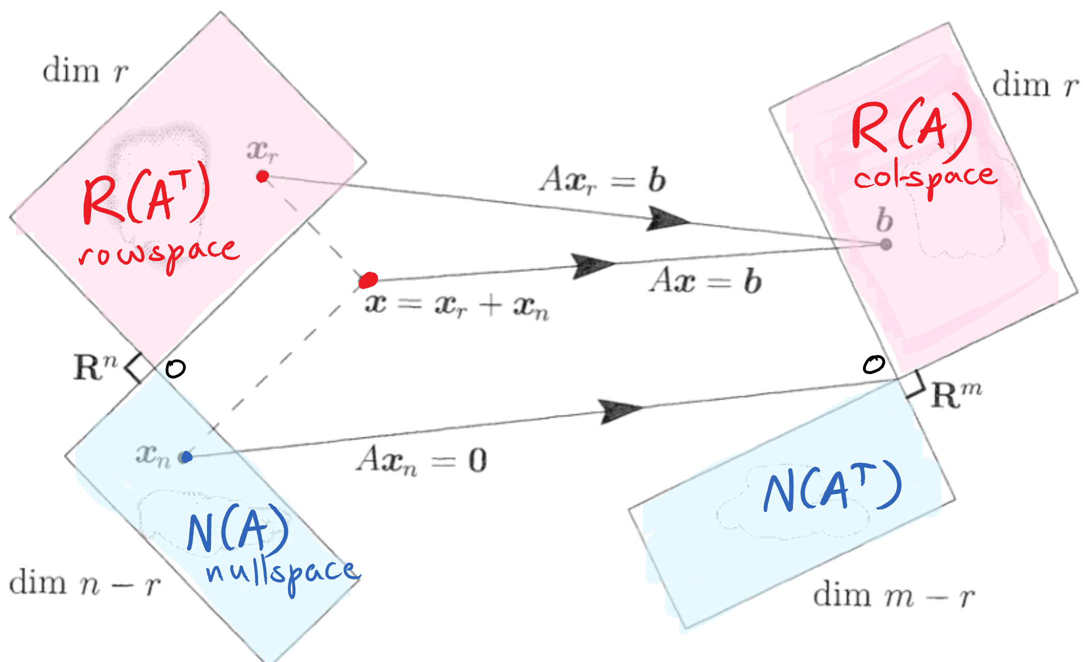
RANK-NULLITY THEOREM





$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\dim N(A) + \dim R(A^T) = \text{total dimension } n} \qquad \boxed{\dim R(A) + \dim N(A^T) = \text{total dimension } m}$$



$$N(A) = R(A^T)^{\perp} \qquad N(A^T) = R(A)^{\perp}$$

Singular-Value Decomposition (SVD)

Informally: Any linear transformation can be split into:
- a rotation, followed by
- scaling vectors in or out

Formally:

Theorem: Any $m \times n$ real matrix A can be written

$$A = \sum_{i=1}^{\min\{m,n\}} \lambda_i \vec{u}_i \vec{v}_i^T$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{\min\{m,n\}} \geq 0$ ← singular values where $\lambda_1, \lambda_2, \ldots, \lambda_{\min\{m,n\}} \geq 0$ ← singular values

$\{\vec{u}_i\}_{i=1,\ldots,m}$ is an orthonormal basis for $\mathbb{R}^m$ ← right singular vectors

$\{\vec{v}_i\}_{i=1,\ldots,n}$ is an orthonormal basis for $\mathbb{R}^n$ ← left singular vectors

Interpretation:

$$A\vec{v}_j = \sum_i \lambda_i u_i (\underbrace{v_i \cdot v_j}_{0 \text{ if } i \neq j}) = \lambda_j \vec{u}_j$$

$\Rightarrow$ A "rotates" $\vec{v}_j$ into $\vec{u}_j$, and scales it by $\lambda_j \geq 0$

Matrix notation:

$$A = \underbrace{\begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}_{m\times m}}_{=\,U} \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}_{m\times n}}_{=\,D} \underbrace{\begin{pmatrix} \text{---} & v_1^T & \text{---} \\ \text{---} & v_2^T & \text{---} \\ & \vdots & \\ \text{---} & v_n^T & \text{---} \end{pmatrix}}_{=\,V^T}$$

U: m×m orthogonal matrix, columns are left singular vectors

D: nonnegative diagonal matrix of singular values

V^T: n×n orthogonal matrix, columns of V are right singular vectors

$$A = U \cdot D \cdot V$$

Spectral Decomposition for normal matrices

Theorem: A has a complete, orthogonal set of eigenvectors

$$\Updownarrow$$

$$A^\dagger A = A A^\dagger. \quad (\text{"A is normal"})$$

Gaussian Elimination

Theorem: (LU Decomposition)

Any $m \times n$ matrix A can be factored as

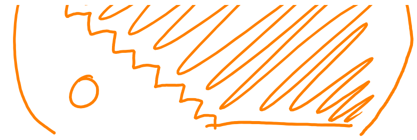
$$\boxed{A = P\,L\,U}$$

permutation matrix $(m\times m)$

lower triangular

upper triangular $m\times n$

(m×m)



Applications:

★ Solving the **same** system of linear equations repeatedly

- Computing Det(A) = Det(P)·Det(L)·Det(U)

MORE REVIEW MATERIAL [Strang]

Matrices	Eigenvalues	Eigenvectors
Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $x_i^{\mathrm{T}} x_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Positive Definite: $x^{\mathrm{T}} A x > 0$	all $\lambda > 0$	orthogonal since $A^{\mathrm{T}} = A$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $x > 0$
Similar: $B = M^{-1} A M$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1} x(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$x = (1, i)$ and $(1, -i)$
Reflection: $I - 2uu^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	u; whole plane $u^{\perp}$
Rank One: uv^{T}	$\lambda = v^{\mathrm{T}} u;\ 0, .., 0$	u; whole plane $v^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi i k/n}$	$x_k = (1, \lambda_k, \dots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$x_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \dots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $J = M^{-1} A M$	diagonal of J	each block gives $x = (0, .., 1, .., 0)$
Rectangular: $A = U\Sigma V^{\mathrm{T}}$	rank(A) = rank(Σ)	eigenvectors of $A^{\mathrm{T}}A, AA^{\mathrm{T}}$ in V, U

LINEAR ALGEBRA IN A NUTSHELL

((*The matrix* A *is* n *by* n))

LINEAR ALGEBRA IN A NUTSHELL

((*The matrix* A *is* n *by* n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x} = \boldsymbol{0}$ has one solution $\boldsymbol{x} = \boldsymbol{0}$	$A\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions
$A\boldsymbol{x} = \boldsymbol{b}$ has one solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$	$A\boldsymbol{x} = \boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

Conceptual review problems [Strang]

§1 Vectors and matrices

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

§2 Solving linear equations

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}$ what are its pivots?

2.4 If A is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \begin{bmatrix} I & U \\ 0 & I \end{bmatrix}$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[A \quad I]$? If you solve the n equations $A\boldsymbol{x} =$ columns of I then the solutions $\boldsymbol{x}$ are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

§3 Vector spaces

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $A\boldsymbol{x} = \boldsymbol{b}$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $A\boldsymbol{x} = \boldsymbol{b}$.

3.9 If $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{b}$, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column $\boldsymbol{u}$ times a row $\boldsymbol{v}^{\mathrm{T}}$ (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^{\mathrm{T}}\boldsymbol{y}$.

3.19 Why do the columns of every invertible matrix yield a basis?

§4 Orthogonality and projections

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If $\boldsymbol{V}$ is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of $\boldsymbol{V}$ and its orthogonal complement add to _____.

4.3 The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector _____.

4.4 The projection matrix onto the line through $\boldsymbol{a}$ is $P =$ _____.

4.5 The key equation to project $\boldsymbol{b}$ onto the column space of A is the *normal equation* _____.

4.6 The matrix $A^{\mathrm{T}}A$ is invertible when the columns of A are _____.

4.7 The least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ minimizes what error function?

4.8 What is the connection between the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection $\boldsymbol{p}$ appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^{\mathrm{T}}Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $\boldsymbol{a} = (2,0)$ and $\boldsymbol{b} = (1,1)$, which two orthonormal vectors does it produce? If we keep $\boldsymbol{a} = (2,0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____.

§5 Determinants

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

§6 Eigenvalues and eigenvectors

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[1\ \ 2\ \ 1]^{\mathrm{T}}[1\ \ 1\ \ 1]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

.9 Explain why the trace of AB equals the trace of BA.

10 How do the eigenvectors of A help to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$?

11 How do the eigenvectors of A help to solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$?

12 Define the matrix exponential e^A and its inverse and its square.

13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^{\mathrm{T}}A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^{\mathrm{T}}A$?

§7 Linear transformations

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from $\boldsymbol{v}$ to $S(T(\boldsymbol{v}))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).