

- Today's Thm wrap up
- Permanent is #P-complete (sketch)
- Space complexity.

Today's Thm: $PH \subseteq P^{\#P}$ aka "counting is powerful"

In fact, this follows from $PH \subseteq BPP^{\oplus P}$ w/ one query, which is what we sketched last time. Where we lost left our heroes (proving $PH \subseteq BPP^{C1D}$), we needed to convert

$$\oplus \Sigma_k \text{-formula} \longrightarrow \oplus \Pi_{k-1} \text{-formula w/ the same truth value.}$$

randomized reduction that succeeds w/ prob. $1 - 1/10k^2$. Then, eventually we'd get some $\oplus$ formula which we'd feed to the oracle.

So, here's the reduction. Start w/

$$\oplus x \exists y \psi(x, y)$$

$$\underbrace{\Pi_{k-1} \text{ formula}}$$

The Valiant-Vazirani proof gives a reduction

$$\exists y \psi(x, y) \longmapsto \exists y \psi(x, y) \wedge "l(y) = 0"$$

so that

$$\exists y \psi(x, y) \Rightarrow \oplus (y, z) (\psi(x, y) \wedge \Theta(z, y))$$

with probability $1/8L$

This success probability is not good enough, we wanted $1/8L^2$ -ish.

So we take the OR of a bunch of these:

$$\bigvee_{i=1}^T \oplus (y, z_i) (\psi(x, y) \wedge \Theta(y, z_i)),$$

and this amplifies ~~the~~ success prob to

$$1 - (1 - 1/8L)^T$$

So now we have

$$\bigvee_{i=1}^T \oplus (y, z_i) (\psi(x, y) \wedge \Theta(y, z_i)) \equiv \exists y \psi(x, y) \quad \forall x \in \{0, 1\}^m \text{ w/ prob } > 1 - \frac{1}{10k^2}$$

By closure properties, we can write $\bigvee_{i=1}^T \oplus (\text{stuff}) = \oplus (\text{stuff})$, and we have our desired reduction.

Permanent is #P-complete (Valiant's Thm)

As we saw last time, $\downarrow$ is ~~an~~ equivalent to:

Thm Counting perfect matchings in a bipartite graph is #P-complete.

Disclaimer: this is a very clever reduction, but it's not clear that you'll ever use this again.

Pt 1. Counting perfect matchings is clearly in #P

To show #P-hardness, we will show

$$\#3SAT \leq \#PM\text{-}BIP\text{-}GRAPH$$

Cycle covers. Given a directed graph, a cycle cover is a partition of its vertices into vertex-disjoint directed cycles.

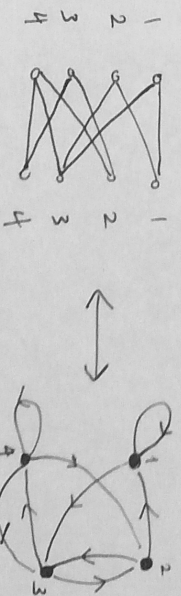


Note that this reduction is NOT parsimonious, else we'd have shown $P = NP$ since deciding PM-BIP is easy.

and #CYCLE-COVER is the problem of counting cycle covers.

#CYCLE-COVER is the source as #PM-BIP

proof by picture:



perfect matching $\iff$ cycle cover



SEP 29 (3)-(5)

To reduce $\#SAT \leq \#CYCLE\ COWER$,
we use some very clever gadgets that are described in
the supplementary notes [posted on Piazza]

SPACE

$SPACE(s(n)) = \{ L \mid L \text{ is decided by a TM w/ space complexity } O(s(n)) \}$

$L = LOGSPACE = SPACE(\log(n))$

$PSPACE = \bigcup_{c \geq 1} SPACE(n^c)$

and $L \in P \subseteq PSPACE \subseteq EXP$.

We also have $PH \subseteq P^{\#P} \subseteq PSPACE$

↓ Total
↓ you can count in
PSPACE (takes a
lot of time, but you
can re-use the space)

~~Consider the language~~

$TQBF = \{ \langle \varphi \rangle : \varphi \text{ is a Totally Quantified Boolean Formula that is TRUE} \}$

a TQBF formula is of the form

$\exists x_1 \forall x_2 \exists x_3 \dots \exists x_n \varphi(x_1, \dots, x_n)$

for example $\exists x_1 \forall x_2 (x_1 \vee x_2)$ is TRUE

$\exists x_1 \forall x_2 (x_1 \wedge x_2)$ is FALSE.

CLAIM $TQBF \in PSPACE$:

pf. Recursively try everything: (depth-first).
You only need to remember $\leq n$ assignments at a time

Thm. $TQBF$ is $PSPACE$ -complete

(under polynomial reduction)
(actually also true under
logspace reductions)

SEP 29 (5)

$$NSPACE(s(n)) = \left\{ L \mid \begin{array}{l} L \text{ decided by } NTM \text{ w/} \\ \text{space } \leq O(s(n)) \end{array} \right\}$$

↳ aka, EVERY branch uses space $O(s(n))$

$$NL = NSPACE(\log(n))$$

$$PSPACE \neq NSPACE(n^c) \quad c \geq 1$$

Three Theorems:

Suppose $s(n) \geq \log(n)$. Then

$$\text{Thm 1. } NSPACE(s(n)) \subseteq \bigcup_c TIME\left(\frac{c \cdot s(n)}{2}\right)$$

CON. $NL \subseteq P$.

- Pf. Let $L \in NSPACE(s(n))$. How to tell if $x \in L$? Consider configuration graph of M on input x , $G_x = (V, E_x)$. This is a directed graph with
- V = all configs of M = (state, ^{write} tape, head positions, ^{write} currently reading)
 - so $|V| \leq 2^{O(s(n))}$

- ~~Fixed state~~
- E_x is given by $\text{State 1} \rightarrow \text{State 2}$ if you can go from State 1 to State 2 on input x .

Thus, we just want to know if there's a path from (START STATE) $\rightarrow$ (ACCEPT STATE) in G_x . We can do this in $\leq$ time $\text{poly}(2^{s(n)})$ ■

$$\text{Thm 2. } NSPACE(s(n)) \subseteq DSPACE((s(n))^2) \quad [\text{SAVITCH'S THM}]$$

$$\text{Thm 3. } NSPACE(s(n)) = \infty - NSPACE(s(n))$$

CON. $NL = \infty NL$.

Next time: pfs of Thms 2 and 3.

Consider the language

$$DIR-PATH = \{ \langle G, s, t \rangle : \begin{array}{l} G \text{ is a directed graph} \\ \text{w/ a directed path} \\ \text{from } s \text{ to } t \end{array} \}$$

Thm (which we'll prove on Thursday) $DIR-PATH$ is NL -complete.

Trivial about:
under what
type of reductions?
polynomial
reductions
don't make
sense.