1

EXPANDER GRAPHS

AGENDA

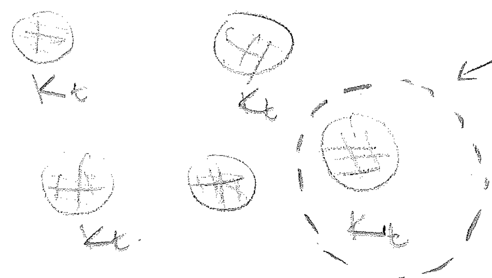
0. Finish hard-core set thm
1. Expander graphs
   - def + examples
   - expander mixing lemma + combinatorial expansion
   - random walks

An expander graph is a d-regular graph that behaves "randomly":

- random walks mix quickly
- two sets have about the "right" # edges between them.
- sets have nbhds of about the "right" size.

Non-examples:

A)

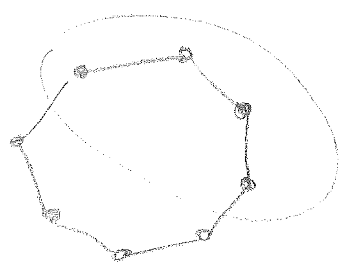


this set does not have the "right" # nbrs, given $deg(G) = t$

- random walks will never mix.

$G = n/t$ copies of K_t

B)

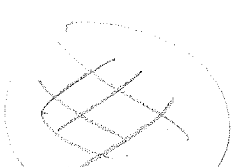


this set should morally have $\Theta(n)$ neighbours, but it only has 2.

- random walk will take a long time to mix.

$G = C_n$

Perfect example:



$G = K_n$

Good example:

random d-regular graph.

How do we quantify this "good" behavior? Linear algebra.

Let $G = (V, E)$ be a d-regular graph on n vertices.

Let A_G be the normalized adjacency matrix:

$$(A_G)_{ij} = \begin{cases} 1/d & (i,j) \in E \\ 0 & \text{else} \end{cases}$$

(so A is symmetric, doubly-stochastic)

Think of A_G as the transition matrix for a random walk on G:

for $p \in \Delta_n \leftarrow \{ (p_1, \ldots, p_n) \in [0,1]^n : \sum_i v_i = 1 \}$

$$\boxed{A_G} \; p = q$$

$\mathbb{P}\{\text{vertex } v\}$
$= \sum_u p_u \mathbb{P}\{v \mid u\}$
$= \sum_u p_u \cdot \frac{1}{d} \cdot \mathbb{1}_{(v,u) \in E}$
$= (A_G p)_v$

Then $q \in \Delta_n$ is the distribution arising from:

1. Pick a vertex according to p
2. Choose one of p's neighbors at random

and $A_G^t p \in \Delta_n$ is the distribution you get from taking a t-step random walk, starting from p.

$\boxed{3}$ Say A_G has eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

Observation: $\lambda_1 = 1$ $\left[\begin{array}{l} A_G \cdot \mathbb{1} = \mathbb{1}, \text{ and} \\ \lambda_1 \leq \max \text{rowsum} = 1 \end{array} \right]$

sanity/linalg check: why are $\lambda_i \in \mathbb{R}$?

<u>Def.</u> $\lambda(G) := |\lambda_2(A_G)| = \max\limits_{\substack{x \perp \mathbb{1} \\ \|x\|_2 = 1}} \|Ax\|_2$

Turns out, $\lambda(G)$ is a good way to quantify what we want.

<u>Examples</u> $G = K_n$, $A = \frac{1}{n}\boxed{\mathbb{1}}$, $\vec{\lambda} = (1, 0, 0, \ldots, 0)$
w/ self loops

$G =$  , $A = \frac{1}{t}\begin{bmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \\ & & \mathbb{1} & \\ & & & \ddots \\ & & & & \mathbb{1} \end{bmatrix}$ $\vec{\lambda} = (\underbrace{1, 1, 1, \ldots 1}_{n/t}, 0, \ldots, 0)$

$\frac{n}{t}$ copies of K_t

$G = C_n$, $A = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & \ddots & \\ 1 & & & & \end{bmatrix}$, $\vec{\lambda} = \left(1, \cos\left(\frac{2\pi}{n}\right), \cos\left(\frac{4\pi}{n}\right), \ldots\right)$

$\approx \left(1, 1 - O\left(\frac{1}{n^2}\right), \ldots\right)$

$G =$ random d regular graph $A = \boxed{\text{random}}$, $\vec{\lambda} \approx \left(1, 1/\sqrt{d}, \ldots\right)$

As this suggests,

λ_2 small $\Rightarrow$ "good"

λ_2 big $\Rightarrow$ not good.

4 Slightly more rigorously,

Lemma $\forall p \in \Delta_n$, $\| A_G^t p - \frac{1}{n}\mathbb{1} \|_2 \le \lambda(G)^t$

aka, if λ is small, a t-step walk converges quickly to uniform.

proof $p = v + \underbrace{\langle p, \mathbb{1}/\sqrt{n} \rangle}_{= 1/\sqrt{n} \text{ since } p \in \Delta_n} \cdot \mathbb{1}/\sqrt{n}$ for $v \perp \mathbb{1}/\sqrt{n}$

$= v + \frac{1}{n}\mathbb{1}$.

Then

$A_G p = A_G\left(v + \frac{1}{n}\mathbb{1}\right) = A_G v + \frac{1}{n} A_G \mathbb{1}$

$= v' + \frac{1}{n}\mathbb{1}$

shorter than $\lambda_2 \|v\|_2 \le \lambda_2$, and still $\perp \mathbb{1}$

$A_G^2 p = A_G v' + \frac{1}{n} A_G \mathbb{1}$

$= v'' + \frac{1}{n}\mathbb{1}$

shorter than $\lambda_2 \|v'\|_2 \le \lambda_2^2$; still $\perp \mathbb{1}$

$\vdots$

$A_G^t p = v^{(t)} + \frac{1}{n}\mathbb{1}$, $\|v^{(t)}\|_2 \le \lambda_2^t$

So Δ-ineq $\Rightarrow$ $\| A_G^t p - \frac{1}{n}\mathbb{1} \|_2 \le \lambda(G)^t$. ∎

SKIP

Cor If λ_2 is small, the diameter of G is small: Given $u, v \in V$

$\mathbb{P}\{\text{random walk of len } t \text{ starting at } u \text{ ends at } v\} = (A_G^t e_u)_v \ge \frac{1}{n} - \lambda(G)^t > 0$

if $t = \Omega(\log(n))$ and $\lambda(G)$ is bounded away from 1.

So $\operatorname{diam}(G) \le \log_{1/\lambda(G)}(n)$

5 How small can we hope $\lambda(G)$ can be? (If $\lambda = 1 - 1/n^2$, like for the cycle, then $\lambda(G)^t$ isn't very small)

- for many purposes, we just want it bdd away from 1
- actually, it can be as small as $\approx 1/\sqrt{d}$ (actually $\frac{2\sqrt{d-1}}{d}$)

such a graph is called "Ramanujan"

- These exist! And we can find them efficiently.

Thm] For infly many d, there are explicit, infinite families of d-regular Ramanujan graphs. [Lubotzky, Phillips, Sarnak] [Morgenstern]

(in particular, we can take d, λ constant and send $n \to \infty$).

So for those graphs, the lemma says that a random walk converges to uniform exponentially quickly.

Other nice properties of random walks on expander graphs:

Thm] Let $B \subseteq V$ have $|B| \leq \beta n$, $\beta \in (0,1)$

$X_0 \in V$ uniformly random

$X_1, X_2, \ldots, X_t$ random walk on G starting from X_0. Then

(1) $\mathbb{P}\{\forall j,\ X_j \in B\} \leq ((1-\lambda)\sqrt{\beta} + \lambda)^t$

(2) (expander chernoff) $\forall \delta > 0$,

$$\mathbb{P}\left\{ \left| \frac{1}{t+1} \sum_{j=0}^{t} \mathbb{1}_{\{X_j \in B\}} - \beta \right| > \delta \right\} \leq 2e^{-(1-\lambda)\delta^2 t/4}$$

(pfs will be on HW)

6. **Application**: Error reduction. Say A is a BPP alg which uses m random bits, and fails w/ prob $1/3$.

Lots of times we've said
"$1/3$ may as well be 2^{-t}"
by repeating a randomized algorithm t times.

This is a pretty wasteful use of randomness:
- we use $t \cdot m$ bits instead of m.

We can do better w/ expanders:

Let G be an expander graph on 2^m vertices (constant d, λ)

Consider the following alg:

$A'(x)$:

Pick r_0 at random in $\{0,1\}^m$
Let $r_1, \ldots, r_t$ be a random walk on G starting from r_0
return Majority$(A(x;r_0), A(x;r_1), \ldots, A(x;r_t))$

This uses $m + t\lg(d)$ random bits.

CLAIM: $\mathbb{P}\{A'(x) \text{ is wrong}\} \le 2^{-\Theta(t)}$.

why? Let $B \subset \{0,1\}^m$ be the set of "bad" seeds for A, so $|B| \le \frac{1}{3} \cdot 2^m$. Then by expander chernoff,

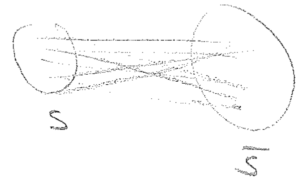
$$\mathbb{P}\{A'(x) \text{ wrong}\} = \mathbb{P}\left\{\left|\frac{1}{t+1}\sum_j \mathbb{1}\{r_j \in B\} - \frac{|B|}{2^m}\right| > \frac{1}{6}\right\}$$
$$\le 2e^{-(1-\lambda)t/c}$$

So we turned $m \cdot t$ into $m + \Theta(t)$, and we still made only t calls to A.

7 Combinatorial notions of expansion

<u>Lemma</u> For $S \subseteq V$, $|E(S,\bar{S})| \geq (1-\lambda)\dfrac{d\,|S|\cdot|\bar{S}|}{n}$

This is "random-like" behavior: If each edge is present w/ prob d/n, we expect $|E(S,\bar{S})|$ to be $\dfrac{d\cdot|S||\bar{S}|}{n}$



More generally.

<u>Expander mixing lemma</u> $\forall\, S,T \subseteq V$ (not nec. disjoint)

$$\Big|\, |E(S,T)| - \tfrac{d}{n}|S||T| \,\Big| \leq \lambda d \sqrt{|S||T|}$$

(proofs ommitted)

(the converse is actually also true:

if $\Big|\, |E(S,T)| - \frac{d}{n}|S||T| \,\Big| \leq \Theta d \sqrt{|S||T|}$ $\forall$ disjoint S,T,

then $\lambda(G) = O\left(\Theta \log\left(\frac{1}{\Theta}\right)\right)$

<u>LEMMA</u> For $S \subseteq V$, $|\Gamma(S)| \geq |S|\cdot\left(\dfrac{1}{(1-|S|/n)\lambda^2 + |S|/n}\right)$

(proof ommitted)

That is, if λ is a small constant, sets have lots of edges coming out and they have big neighbourhoods.

$\boxed{7.5}$ Another way to do error reduction (via expander mixing lemma)

Again, let A be a BPP alg w/ m random bits.

Now, I can use JUST m random bits and do a bit better:

$A'(x)$:

Choose $r_0 \in \{0,1\}^m$ at random

Return majority$\{A(x;r) : (r, r_0) \in E(G)\}$

Analysis:

Let $B \subseteq \{0,1\}^m$ be "bad" seeds for A

Let $B' \subseteq \{0,1\}^m$ be "bad" for A'

Notice that $r \in B' \Rightarrow r$ has at least $d/2$ nbrs in B, so

$$|E(B',B)| \geq |B'| \cdot \frac{d}{2}$$

OTOH, EML says

$$\frac{d}{2}|B'| \leq |E(B',B)| \leq \frac{d}{2^m}|B'|\cdot|B| + d\lambda\sqrt{|B'||B|}$$

solving for $|B'|$, get

$$|B'| \leq |B| \cdot \left(\frac{\lambda^2}{1/2 - |B|/2^m} \right) \leq \frac{2^m}{3} \cdot \frac{\lambda^2}{1/6} \leq 2 \cdot 2^m \cdot \lambda^2$$

so if G is Ramanujan,

$$\mathbb{P}\{A' \text{ fails}\} = \frac{|B'|}{2^m} \leq 2\lambda^2 \leq O(1/d).$$

(this is a slightly more efficient use of randomness: m instead of $m + \lg^2(d)$ but a less efficient use of time: d runs of A instead of $\log(d)$ runs of A.)

Some more combinatorial corollaries of spectral expansion:

(1) All independent sets in G are smaller than $\lambda(G)\cdot n$

pf) If S indep set, $E(S,S)=0$, so EML says

$$\left|0-\tfrac{d}{n}|S|^2\right| < \lambda d|S| \Rightarrow |S| \le \lambda n$$

(2) Chromatic # $\chi(G)$ is $\geq 1/\lambda(G)$

pf) all vertices of the same color are an indep set, so

$$n \le \chi(G)\cdot \lambda n \Rightarrow \chi(G) \ge 1/\lambda$$

(3) diameter of G is $\le \log_{1/\lambda}(n)$

pf) For any $u,v\in V$, $\mathbb{P}\{$walk starting at u of len t ends at $v\}$

$$= (A_G^t e_u)_v \ge \frac{1}{n} - (\lambda(G))^t \gneq 0$$

if $t \ge \log_{1/\lambda}(n)$.

So there is a path of len $\approx \log_{1/\lambda}(n)$ from u to v.

Expander graphs are very useful! We'll see some more uses next week when we talk about ... extractors!