Last Class:

Smolesky's pf that PARITY $\notin AC^0$

Approximating Boolean fcns by polynomial $\Rightarrow$ Rich topic with many applications

Switching Lemma

k-DNF, $f: \{0,1\}^n \to \{0,1\}$

hit it with a fraction p random restriction ρ. $f|_\rho : \{0,1\}^{pn} \to \{0,1\}$

$Pr[\cancel{k}\ f|_\rho \text{ is not a } k\text{-CNF}] \leq (7pk)^t \quad t = O(\log S)$

Apply the Switching Lemma for d-2 times, the free variables # $= \frac{n}{(\log S)^{d-2}}$

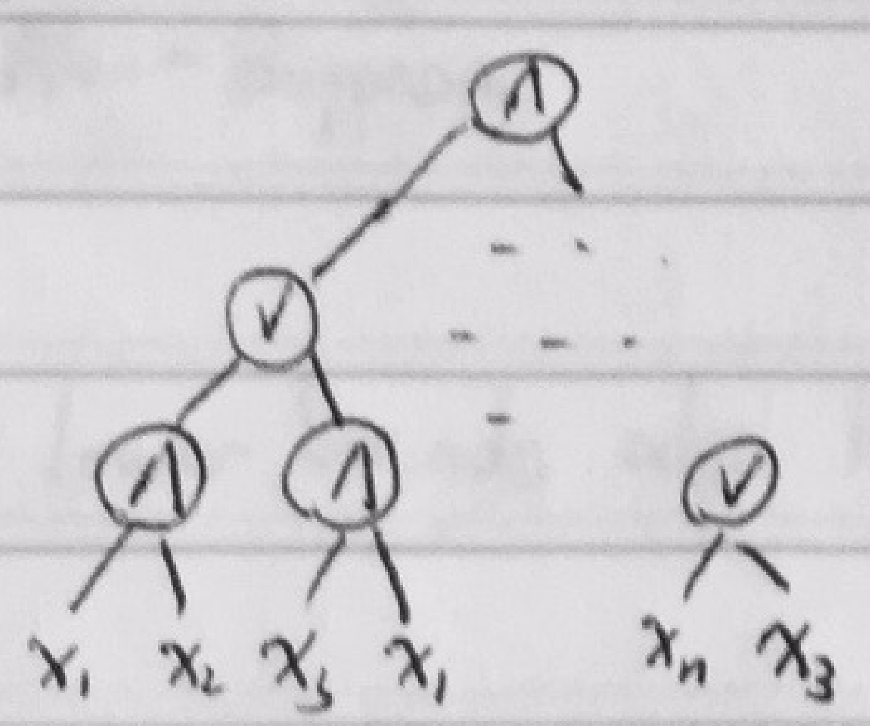
$S \geq 2^{\Omega(\frac{n}{(\log S)^{d-2}})} \to$ exp. lower bound for depth 2

$\log S \geq \Omega(\frac{n}{(\log S)^{d-2}})$

$S \geq 2^{\Omega(\frac{n}{d-1})}$

Today: Monotone Circuits

Monotone Circuits: $\vee, \wedge$ gates with fan-in 2, no negations



Monotone Circuits can only compute Monotone fcns

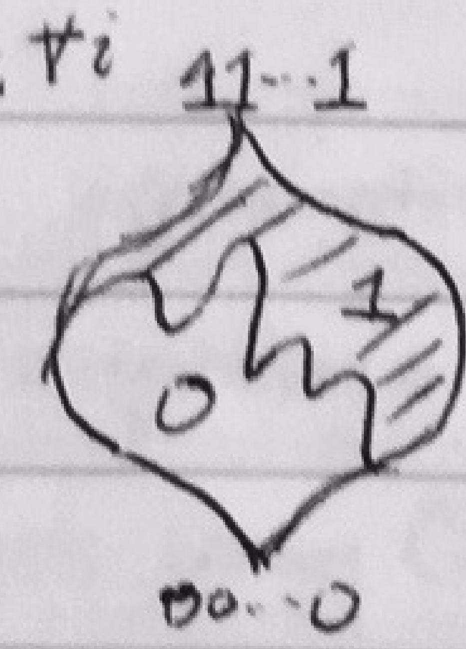
Def: $f: \{0,1\}^n \to \{0,1\}$ is monotone if

$x \leq y \Rightarrow f(x) \leq f(y)$

$\Updownarrow$

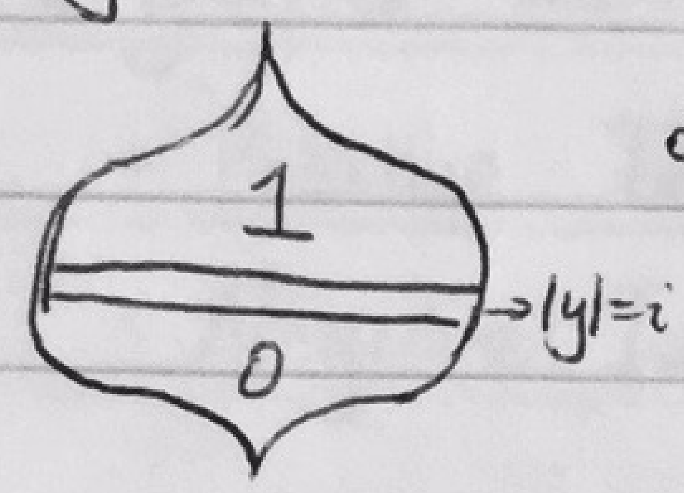
$x_i \leq y_i \ \forall i$

Illustration:



f is monotone $\Leftrightarrow \exists$ a monotone ~~funct~~ circuit computes f

If g is not monotone,



define $g_i(y) = \begin{cases} g(y) & |y|=i \\ 1 & |y|>i \\ 0 & |y|<i \end{cases}$ $\qquad g(y) = \bigvee_{i=0}^{n} \left((|y|=i) \wedge g_i(y) \right)$

So lower bound for monotone circuits $\Rightarrow$ lower bounds for all circuits

Majority fcns is monotone and has poly(n) sized monotone formula.

We can write a property of graph as a function $f: \{0,1\}^{\binom{n}{2}} \to \{0,1\}$

$x_e = 1 \Rightarrow e \in E(G) \quad x_e = 0 \Rightarrow e \notin E(G)$

$CLIQUE_{n,k} : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$

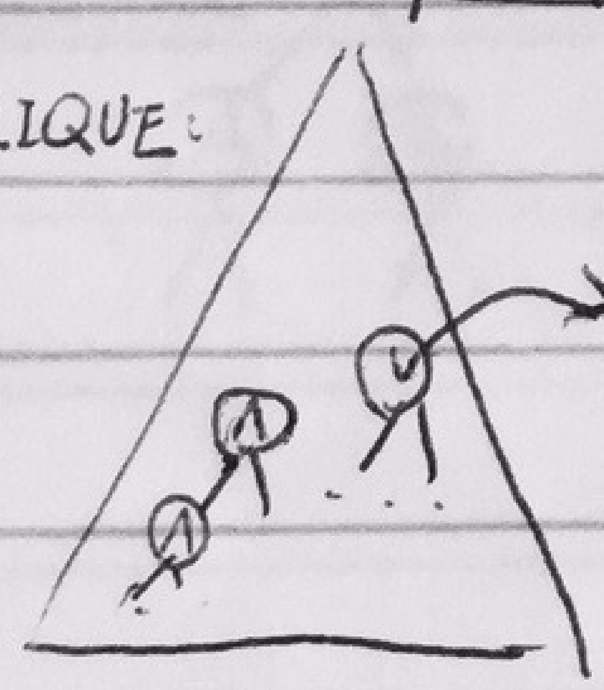
$CLIQUE_{n,k}(G) = \begin{cases} 1 & \text{if } \omega(G) \geq k \\ 0 & \text{if } \omega(G) < k \end{cases}$ $\rightarrow$ clique number

Thm [Razborov '85] $CLIQUE_{n,k}$ requires monotone circuits of size $n^{\Omega(\sqrt{k})}$
+ Andreev $\qquad$ for $n^{\frac{1}{8}} \leq k \leq n^{\frac{1}{4}}$
+ Alon-Boppana

Alas, lower bounds also holds for functions in P. (e.g. Perfect-Matching)

Method of Approximation

CLIQUE:



$\rightarrow$ Approximate each gate

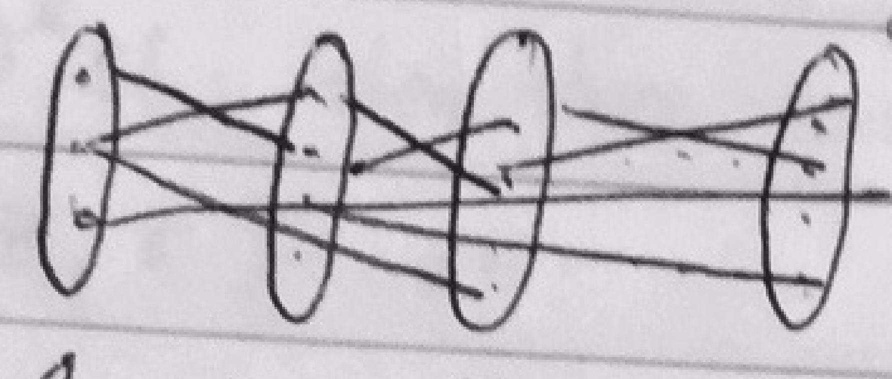
① After approximating all gates, final computations is "too simple" s.t. gets ~~wrong~~ CLIQUE wrong on a lot of places

② At each gate, the approximating operation introduces few new wrong cases.

① + ② $\Rightarrow$ Lots of gates!

Tests graphs we'll use in error analysis:

- Positive Tests: k-Clique & nothing else $\# = \binom{n}{k}$ graphs
- Negative Tests: $(k-1)$-colorable graphs



1 2 3 $k-1$ (No edge between nodes with the same color)

Complete $(k-1)$-partite graphs: $\# = \cancel{\frac{(k-1)^n}{(k-1)!}}\ (k-1)^n$

(Consider different coloring as different graphs)

<u>Approximation</u>

<u>Def</u>: Clique indicator $I_X = \begin{cases} 1 & \text{if } X \text{ induces } \cancel{\text{indicates}} \text{ a clique} \\ 0 & \text{if } X \text{ not} \end{cases}$

~~Replace each gate by~~ a (m, l)-approximator:

$I_{X_1} \vee I_{X_2} \vee \cdots \vee I_{X_r}$

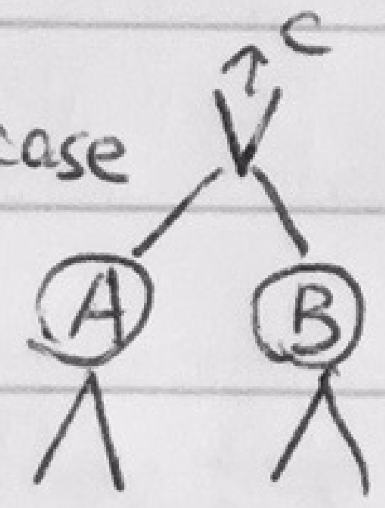
① $r \leq m$ ② Each $|X_i| \leq l$

(m, l)-approximator is OR of at most m clique indicators, each of which has size $\leq l$

Building the approximate "circuit" (m, l)-approximator

For leaves: $(1,1)$-approximator

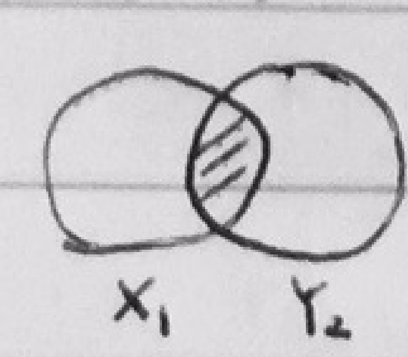
OR case



: $A = \bigvee_{i=1}^{r} I_{X_i}$ $\quad C = A \vee B$

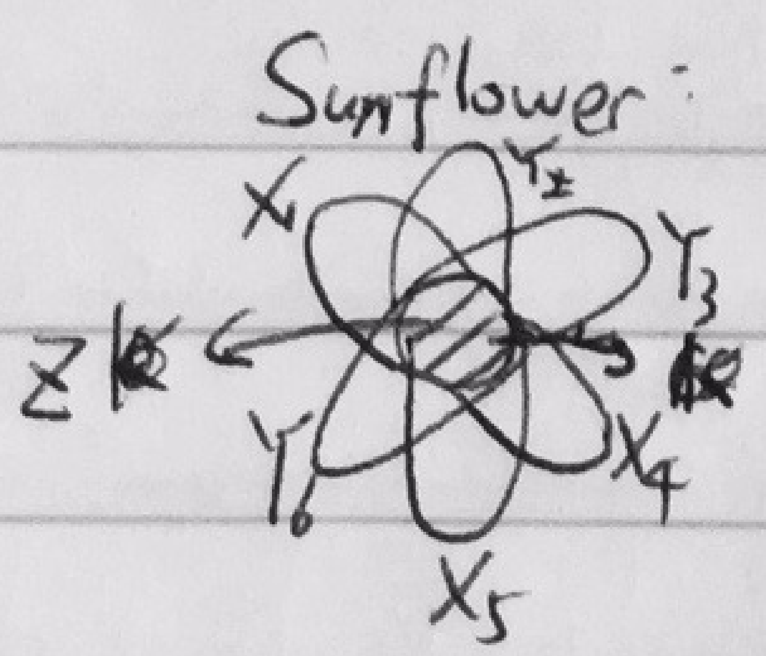
$B = \bigvee_{j=1}^{s} I_{Y_j}$ $\quad (2m, l)$-approximator $\Rightarrow (m, l)$-approximator

$|X_i|, |Y_j| \leq l$



$I_{X_1} \vee I_{Y_2} \rightarrow I_{X_1 \cap Y_2}$

Sunflower:



$I_{X_1} \vee I_{Y_2} \vee I_{Y_3} \vee I_{X_4} \vee I_{X_5} \vee I_{Y_6}$

$\rightarrow I_Z$

We introduce another parameter: $p = \#$ ~~pedals~~ petals

Sunflower Lemma

Suppose a family $\mathcal{F} \subseteq 2^{[n]}$ s.t.

$\forall S \in \mathcal{F}$, $|S| \leq \ell$.

If $|\mathcal{F}| > (p-1)^\ell \ell!$, then there exists a "p-petalled sunflower"

i.e. $\exists Z_1, \cdots, Z_p \in \mathcal{F}$ s.t.

$$Z_i \cap Z_j = Z \quad \forall i \neq j$$

Pf is by induction.

$(2m, \ell)$-approximator $\to$ (m, ℓ)-approximator:

Repeat

- Find sunflower
- pluck its petals (Replace $I_{Z_1} \vee \cdots \vee I_{Z_p} \to I_Z$)

Until we get down to m sets

AND Case: $A \wedge B = \left(\bigvee_{i=1}^{r} I_{X_i}\right) \wedge \left(\bigvee_{j=1}^{s} I_{Y_j}\right) = \left(\bigvee_{i=1}^{r} \bigvee_{j=1}^{s} (I_{X_i} \wedge I_{Y_j})\right)$

Replace $I_{X_i} \wedge I_{Y_j} \to I_{X_i \cup Y_j}$, drop it if $|X_i \cup Y_j| > \ell$

Use the same process as in ~~A~~ OR case to pluck

(m^2, ℓ)-approximator to (m, ℓ)-approximator.

For using Sunflower Lemma, we set $m = (p-1)^\ell \ell!$

Therefore the whole circuit is a (m, ℓ)-approximator.

~~Lemma~~:

(m, ℓ)-approximator C' at output gates makes lots of errors.

Lemma:

Either C' is wrong on all positive cases, or it is wrong on at least $\left(1 - \frac{1}{k-1}\binom{\ell}{2}\right)$ frac of neg cases.

(Pick $\ell \approx \sqrt{k}$, wrong probability $\geq \frac{1}{2}$)

Pf:

C' is wrong on all pos cases $\Rightarrow C' \equiv 0$

If not, $C = \bigvee_{i=1}^{r} \lceil X_i$, $0 < r \leq m$

$\lceil X_1 = 1 \Rightarrow$ All vertices in X_1 is a clique, $|X_1| \leq \ell$.

$$\Pr_{\substack{\text{random} \\ \text{neg cases}}} [\lceil X_1 = 0] \leq \frac{\binom{\ell}{2}}{k-1}$$

Pf of Sunflower Lemma:

Basic case: $\ell = 1$, pick p sets with empty core

$\ell > 1$:

Pick a maximal collection of disjoint sets $Z_1, \cdots, Z_q$.

If $q \geq p$, we are done with empty core

If $q \leq p-1$, consider a set $Z^* = \bigcup_{i=1}^{q} Z_i$, any other set $Z \in \mathcal{F}$ intersect with Z^*

$\exists z \in Z^*$ that is contained in $\geq \frac{(p-1)^\ell \ell!}{(p-1)\ell} = (p-1)^\ell (\ell-1)!$ sets

Then by inductive hypothesis,

$$\exists Z_1, Z_2, \dots, Z_p \in \mathcal{F}, \quad Z_i \cap Z_j = Z \cup \{z\}, \ \forall i \neq j$$