Lecture 23: Hardness in P

Outline:

- Fine-grained reductions
- 3SUM requires $\tilde{\Omega}(n^2)$ time $\Rightarrow$ Geometry Problems
- All pair shortest path requires $\tilde{\Omega}(n^3)$ time $\Leftrightarrow$ Dense Graph Problems
- ETH, SETH $\Rightarrow$ Orthogonal Vectors require $\tilde{\Omega}(n^2)$ time $\Rightarrow$ Sparse Graph Problems/LC.

Recall NP-complete problems:

1. Identify key hard problems (3SAT/SAT)
2. Reduce these to other problems (TSP, Clique, Vertex cover)
3. Form some equivalent classes (NP-complete)

Two popular conjecture on 3SAT:

ETH: 3-SAT requires $2^{\delta n}$ time, for some $\delta > 0$

SETH: For every $\varepsilon > 0$, there is a k such that k-SAT on n viables, m clauses cannot be solved in time $2^{(1-\varepsilon)n}$ poly m times.

In theoretical CS, poly time = efficient/easy.
but it might be not in practice.
Though a lot of problems (MST, single source shortest paths, min cut) are close to linear, on many problems we stuck.

- No $O(n^{2-\varepsilon})$ time alg for many problems in computational geometry.
  - 3-points-in-line: Given n points on a plane, are any 3 in a line?
- No $O(n^{2-\varepsilon})$ time alg for many ~~problems in~~ string matching problems:
  - Edit distance, Longest Common subsequence.
- No $O(|V|^{3-\varepsilon})$ time alg known for many problems in dense graphs
  sparse
  $O(|V|^{2-\varepsilon})$ ~~tim~~

  Diameter, Radius, Median

  $\max_{x,y} d(x,y)$ $\min_x \max_y d(x,y)$ $\min_x \sum_y d(x,y)$

Are we stuck because of the same reason?

Key Problems for hardness in P

3SUM: Given a set S of n integers, are there any $a,b,c \in S$ with $a+b+c=0$?

All pairs shortest paths (APSP): given a weighted graph, find the ~~de~~ distance between every two nodes

Orthogonal Vectors: Given a set S of n vectors in $\{0,1\}^d$, for $d=O(\log n)$ are there $u,v \in S$ with $u \cdot v = 0$?

3SUM Conj: 3SUM requires $n^{2-o(1)}$ time

APSP Conj: APSP requires $n^{3-o(1)}$ time

OV Conj: OV requires $n^{2-o(1)}$ time.

Fine-grained reduction:

A is (a,b)-redutible to B if:

for every $\varepsilon > 0$, $\exists \delta > 0$, and an $O(a(n)^{1-\delta})$ alg. that transform any A-instance with size n to B instances of size $n_1, \cdots, n_k$, so that $\sum_{i=1}^{k} b(n_i)^{1-\varepsilon} < a(n)^{1-\delta}$.

If B can be solved in $b(n)^{1-\varepsilon}$ time, A can be solve in $a(n)^{1-\delta}$ time.
We can build equivalence.

Structures in P:

3SUM $(n^{2-\varepsilon}) \Rightarrow$ Geometry problems (Geombase, 3PointsLine, 3LinePoints) $(n^{2-\varepsilon'})$

APSP $(n^{3-\varepsilon}) \Leftrightarrow$ In dense graphs (radius, median, negative triangle) $(n^{3-\varepsilon})$

SETH $\Rightarrow$ OV $(n^{2-\varepsilon}) \Rightarrow$ Sparse graph diameter, string problems $(n^{2-\varepsilon'})$

3SUM: given n integers, do any 3 sum to 0? [Gajentaan & Overmars '95]

- conjecture: no $O(n^{2-\varepsilon})$ alg
- easy $O(n^2)$ randomized alg.
  - compute all pairwise sums, check the hashing table
- easy deterministic alg.
  - presort integers.
  - for each target sum:
    - shift left pointer right if too small, shift right pointer left if too big.
- $O(n + u \lg \lg u)$ via FFT if integers $\in [-u, u]$
- ~~$O(n^2)$ when $u = n^3$~~ 3SUM $\Rightarrow$ 3SUM with $u = n^3$.

Distinct 3SUM: $\exists$ 3 ~~SUM~~ distinct integers summing to 0?

Thm: 3SUM $\Rightarrow$ Distinct 3SUM (Check for doubled/tupled integers)

3SUM': Given set A, B, C of n integers, $\exists\, a \in A,\ b \in B,\ c \in C$ s.t. $a + b = c$?

Thm: 3SUM $\Leftrightarrow$ 3SUM'

Pf: "$\Rightarrow$" set $A = B = S$, $C = -S$

"$\Leftarrow$" Assume all $a, b, c \in [0, u]$

set $S = \{A + 2u\} \cup \{B \cup (-C - 2u)$

GeomBase: Given n points in 2D with $y \in \{0, 1, 2\}$,

$\exists$ nonhorizontal line hitting 3 points?

Thm: 3SUM' $\Leftrightarrow$ GeomBase

Pf: $a \in A \leftrightarrow (a, 0)$ $\qquad a + b = c \Leftrightarrow (a,0), (b,2), (\frac{c}{2}, 1)$ in a line

$b \in B \leftrightarrow (b, 2)$

$c \in C \leftrightarrow (\frac{c}{2}, 1)$

3-Points-On-line: Given a set of points in plane, any 3 collinear?

Thm: Distinct 3SUM $\Rightarrow$ 3-Points-On-line

Pf: Set points (x, x^3) $\forall x \in S$

APSP:

- $O(n^3/\sqrt{\log n})$ [Ryan Williams '14]

APSP $\Longleftrightarrow$ Distance product $\Longleftrightarrow$ Neg. Triangle   [V,W, AGV'14]

Median   Radius

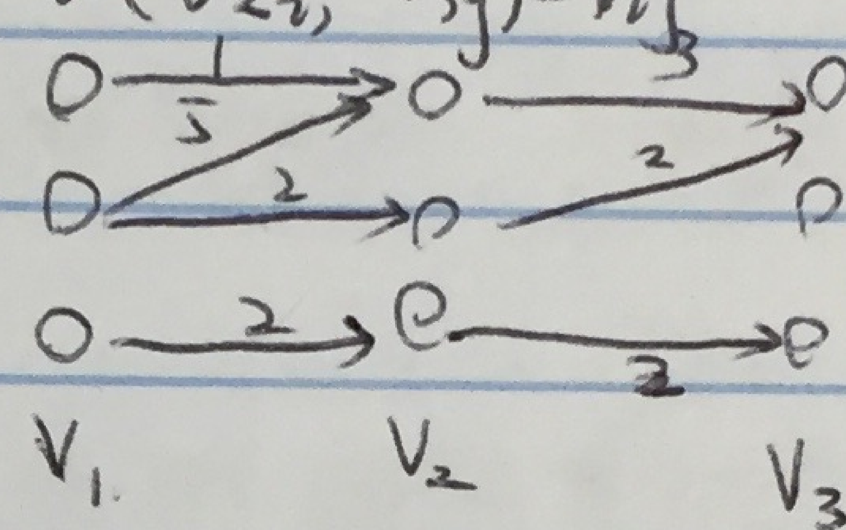
Distance Product: Given two matrix A, B, calculate $(A * B)_{ij} = \min_k (A_{ik} + B_{kj})$

Thm: APSP $\Leftrightarrow$ Distance Product

Pf "$\Rightarrow$"   Set $V = V_1 \cup V_2 \cup V_3$, $|V_1| = |V_2| = |V_3| = n$.

$w(v_{1i}, v_{2j}) = A_{ij}$ $w(v_{2i}, v_{3j}) = B_{ij}$

$A \begin{bmatrix} 1 & \infty & \infty \\ 5 & 2 & \infty \\ \infty & \infty & 2 \end{bmatrix}$



$B \begin{bmatrix} 3 & \infty & \infty \\ 2 & 2 & \infty \\ \infty & \infty & 2 \end{bmatrix}$

"$\Leftarrow$" $A_{uv} = \begin{cases} w(u,v) & \text{for } (u,v) \in E \\ \infty & \text{for } (u,v) \notin E \text{ and } u \neq v \\ 0 & \text{for } u = v. \end{cases}$

A^n is the shortest paths for all pairs.

Negative triangle: $\exists\, i,j,k \in V$, $w(i,j) + w(j,k) + w(k,i) < 0$?

All pairs negative triangle: $V = V_1 \cup V_2 \cup V_3$ ~~$I \times J \times K$~~, $|I| = |J| = |K| = n$, output matrix ~~A~~ C

$C_{ij} = \begin{cases} 1 & \text{if } \exists k \in \cancel{K},\ \cancel{w(i,j) + w(j,k) + w(k,i) < 0}\ \ w(v_{1i}, v_{2k}) + w(v_{2k}, v_{3j}) + w(v_{3j}, v_{1i}) < 0 \\ 0 & \text{otherwise} \end{cases}$

Thm: Distance Product $\Rightarrow$ All pairs Negative Triangle $\Rightarrow$ Negative Triangle

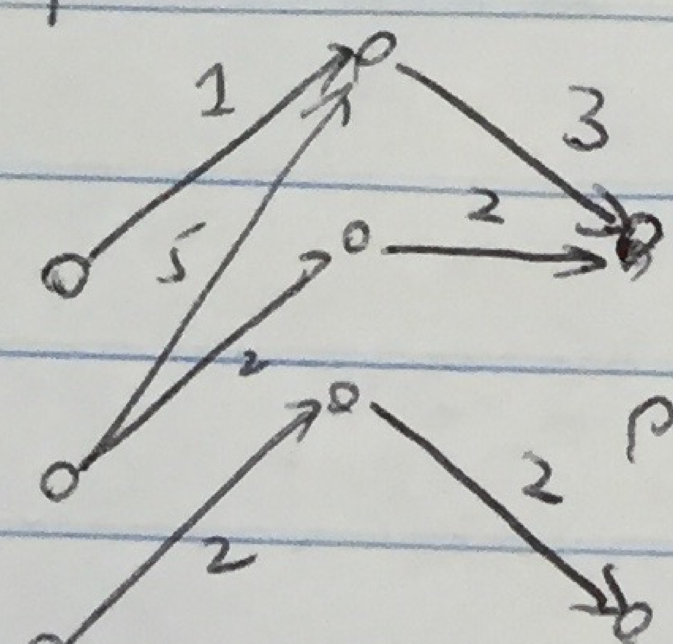
Pf: Distance Product $\Rightarrow$ All pairs Negatie Triangle.

Set $w(v_{1i}, 2j) = A_{ij}$. $w(v_{2i}, v_{3j}) = B_{ij}$

Add edges from V_3 to V_1 with carefully chosen weights

Use APNT for simultaneous binary search

$$\begin{matrix} 1 & \infty & \infty \\ 5 & \cancel{\infty}\,2 & \infty \\ \infty & \infty & 2 \end{matrix}$$



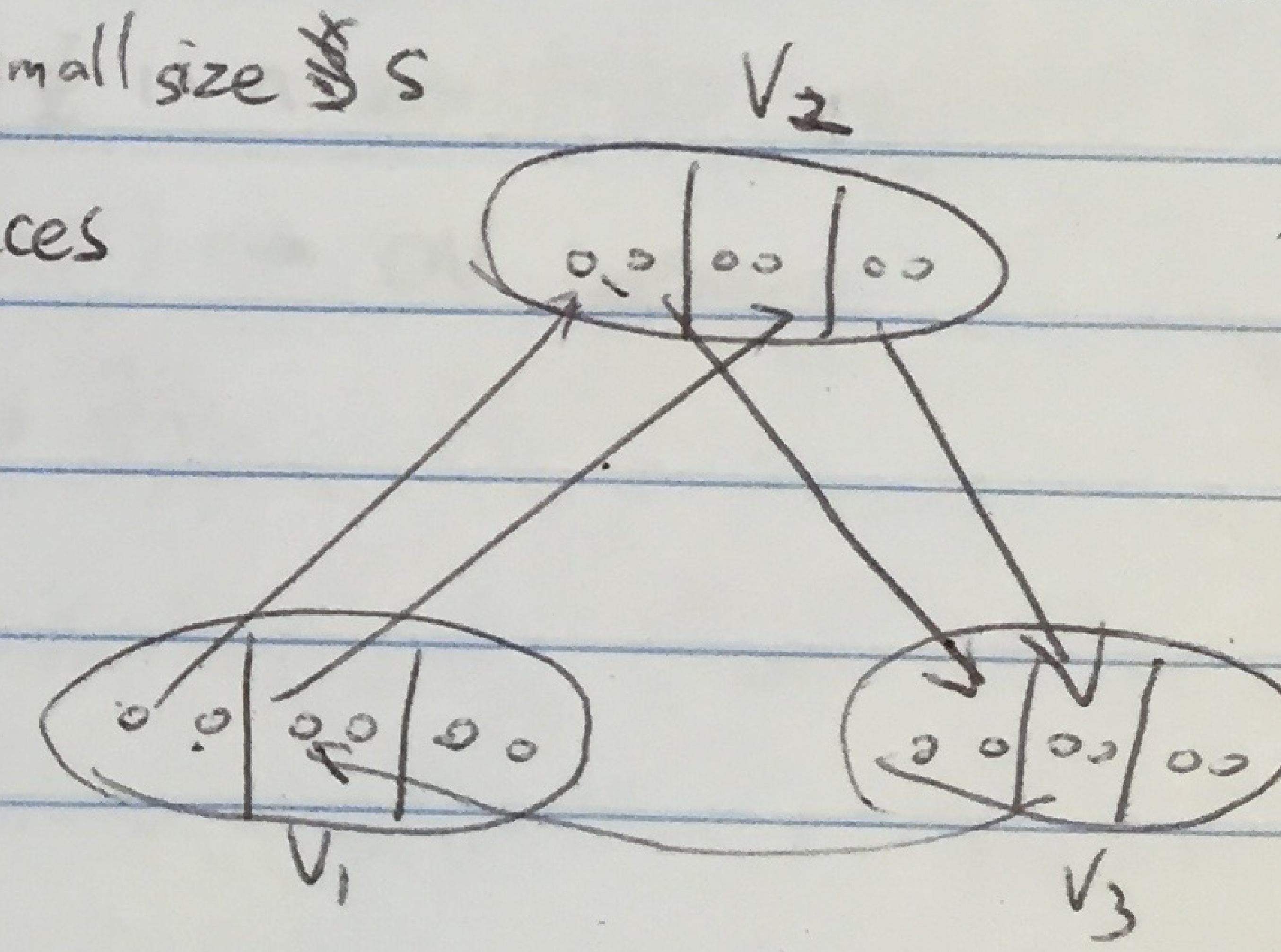
$$\begin{matrix} 3 & \infty & \infty \\ 2 & \infty & \infty \\ \infty & \infty & 2 \end{matrix}$$

$\exists$ $w(v_{3j}, v_{1i}) < -(A*B)_{ij} \Leftrightarrow C_{ij} = 1$

All pairs Negatie Triagle $\Rightarrow$ Negative Triangle.

Ideas:

- ~~Consider~~ Split V_1, V_2, V_3 into pieces of small size s
- Consider all ~~pairs~~ $(n/s)^3$ triples of pieces
- Find negative triangles in each triple



Alg:

Initialize C: $n \times n$ metrix of all 0s.

For every V_{1x}, V_{2y}, V_{3z} in turn:

while (V_{1x}, V_{2y}, V_{3z}) has a negative triangle:

~~report negative triangle a_{1x}, a_{2y}, a_{3z}~~

set $C(a_x, a_z) = 1$

set $w(V_{1a_z}, V_{1a_x}) = \infty$

Runtime:

$[(\#\text{triples}) + (\#\text{triangles fond})] \cdot T(s)$

$= [(n/s)^3 + n^2]\, T(s)$

$= n^{2+d/3}$ $(T(s) = s^d,\ s = n^{1/3})$

~~Subqu~~ Subcubic if $d < 3$

Orthogonal Vectors:

Thm: SETH $\Rightarrow$ OV Conjecture

Pf: Fast OV implies SETH is false.

By sparsification lemma, any k-CNF can be converted into a small number of k-CNFs on $O(n)$ clauses.

For k-CNF on n variables, $m = O(n)$ clauses.

Split V into V_1 and V_2, $|V_1| = |V_2| = n/2$

For $j = 1, 2$, and any partial assignment ϕ of V_j, create $[m+2]$ length vectors

$v[j, \phi]$:

m clauses

0	1	0	1	0	- - - - - -	1

$j=1$ — 0 if ϕ satisfies the clause, 1 otherwise

1	0	1	0	1	- - - - - -	1

$j=2$ — 0 if Ψ satisfies the clause, 1 otherwise

Claim: $v(1, \phi) \cdot v(2, \Psi) = 0$ iff $\phi \cup \Psi$ is a satisfiable assignment

$N = 2^{n/2}$ vectors of length $n = O(\log N) \rightarrow$ OV instance

So $O(N^{2-\delta})$ time on OV $\Rightarrow$ SETH is false.