What is Logic?

Definition

Logic is the study of valid reasoning.

- Philosophy
- Mathematics
- Computer science

Definition

Mathematical Logic is the mathematical study of the methods, structure, and validity of mathematical deduction and proof. [Wolfram Mathworld]

Propositions

Definition

A *proposition* is a declaritive sentence that is either true or false.

- Today is Tuesday.
- Today is Wednesday.
- $5 + 2 = 7$
- $3 \cdot 6 > 18$
- The sky is blue.
- Why is the sky blue?
- Barack Obama.
- Two students in the class have a GPA of 3.275.
- The current king of France is bald.

Conjunction

Definition

The *conjunction* of two propostions, p and q, is the proposition "p and q". It is true when both p and q are true.

Example

s: The sky is blue.

g: The grass is green.

m: The moon is made of cheese.

$s \wedge g$: The sky is blue and the grass is green.

$s \wedge m$: The sky is blue and the moon is made of cheese.

Disjunction

Definition

The *disjunction* of two propostions, p and q, is the proposition "p or q". It is true when either p or q is true.

Example

s: The sky is blue.
g: The grass is red.
m: The moon is made of cheese.
$s \vee g$: The sky is blue or the grass is red.
$g \vee m$: The grass is red or the moon is made of cheese.

Truth tables . . .

The meaning of a logical operation can be expressed as its "truth table."

- Construct the truth-table for conjunction.
- Construct the truth-table for disjunction.
- Construct the truth-table for negation.

Do in class.

A worked example

Example

Let s be "The sun is shining" and t be "It is raining." Join these into the compound statement:

$$(\neg s \wedge t) \vee \neg t.$$

- Phrase the compound statement in English.
- Construct the truth table.

Do in class.

Exclusive or

The word "or" is often used to mean "one or the other," but this is *not* the same meaning of "or" in logic!

Definition

The *exclusive-or* of two statements p and q (written $p \oplus q$), is true when either p is true or q is true, but not both.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical equivalences

How do we know if two logical statements are equivalent?

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- Construct truth tables for each.
- Check if final columns match.

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Theorem

Let p and q be statement variables. Then

$$(p \vee q) \wedge \neg(p \wedge q) \equiv p \oplus q$$

$$\text{and} \quad (p \wedge \neg q) \vee (q \wedge \neg p) \equiv p \oplus q \ .$$

Prove in class (using Truth Tables).

Conditional Statements

Hypothesis $\longrightarrow$ Conclusion

Example

- If it is raining, I will carry my umbrella.
- If you don't eat your dinner, you will not get dessert.

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Expressing Conditionals

Conditional can be expressed in many ways:

- if p then q
- p implies q
- q if p
- p only if q
- a sufficient condition for q is p
- a necessary condition for p is q

More on Conditional

In logic the hypothesis and conclusion need not relate to each other.

Example

- If Joe likes cats, then the sky is blue.
- If Joe likes cats, then the moon is made of cheese.

In programming languages “if-then” is a command.

Example

- If it rains today, then buy an umbrella.
- If $x > y$ then $z := x + y$

Four important variations of implication

- Contrapositive
- Converse
- Inverse
- Negation

Contrapositive

Definition

The *contrapositive* of a conditional statement is obtained by transposing its conclusion with its premise and inverting. So,

$$\text{Contrapositive of } p \rightarrow q \text{ is } \neg q \rightarrow \neg p.$$

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Contrapositive: *If I don't live in Maryland, then I don't live in College Park.*

Theorem

The contrapositive of an implication is equivalent to the original statement.

Prove in class.

Converse

Definition

The *converse* of a conditional statement is obtained by transposing its conclusion with its premise.

Converse of $p \longrightarrow q$ is $q \longrightarrow p$.

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Converse: *If I live in Maryland, then I live in College Park.*

Inverse

Definition

The *inverse* of a conditional statement is obtained by negating both its premise and its conclusion.

Inverse of $p \rightarrow q$ is $(\neg p) \rightarrow (\neg q)$.

(Parentheses added for emphasis.)

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Inverse: *If I don't live in College Park, then I don't live in Maryland.*

The inverse of an implication is equivalent to the converse!

Why?

Negation

Definition

The *negation* of a conditional statement is obtained by negating it.
Negation of $p \to q$ is $\neg(p \to q)$ (which is equivalent to $p \wedge \neg q$).

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Negation: *I live in College Park, and I don't live in Maryland.*

The negation of a conditional statement is not a conditional statement!

Biconditional Statements

Example

- I will carry my umbrella, if and only if it is raining.
- You will get dessert, if and only if you eat your dinner.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Expressing Biconditionals

Biconditional can be expressed in many ways:

- p iff q
- p is necessary and sufficient for q
- p is a necessary and sufficient condition for q

Experimenting with biconditionals

Questions:

- What do the converse, inverse, and negations of a bi-conditional look like?
- What is the relationship between the exclusive-or (discussed above) and the bi-conditional?

Laws of Propositional Logic . . .

We can do algebra in propositional logic.

Commutative Laws:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

Associative Laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive Laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

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Prove a Distributive Law in class.

De Morgan's laws ...

Theorem (De Morgan's laws)

Let p and q be statement variables. Then

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\text{and} \quad \neg(p \wedge q) \equiv \neg p \vee \neg q \;.$$

Examples in English

Example

It is not the case that Alice or Bob went to the store.
$\equiv$ Alice did not go to the store and Bob did not go to the store.

It is not the case that Alice and Bob went to the store.
$\equiv$ Alice did not go to the store or Bob did not go to the store.

Prove in class (using Truth Tables).

Laws of Logic

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold:		
1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double Negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. DeMorgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Universal bounds laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

Example of Boolean Algebra

$$\neg(\neg p \land q) \land (p \lor q) \;\equiv\; p$$

Prove in class (using Boolean algebra).

Logic and Bit Operations

Do in class.