Section 13.2, #56. Find the location and velocity at t = 4 of a partical whose path satisfies

$$\frac{d\mathbf{r}}{dt} = \left\langle 2t^{-1/2}, 6, 8t \right\rangle, \quad \mathbf{r}(1) = \langle 4, 9, 2 \rangle$$

Solution. The velocity of the particle is exactly:

$$\mathbf{r}'(4) = \left\langle 2(4)^{-1/2}, 6, 8(4) \right\rangle = \langle 1, 6, 32 \rangle$$

To determine the location of the particle at any general t, we will perform integration componentwise on $\mathbf{r}'(t)$:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)dt = \int \left\langle 2t^{-1/2}, 6, 8t \right\rangle dt = \left\langle 4\sqrt{t}, 6t, 4t^2 \right\rangle + \mathbf{c_1}$$

Using the initial condition:

$$\mathbf{r}(1) = \langle 4, 6, 4 \rangle + \mathbf{c_1} = \langle 4, 9, 2 \rangle$$

$$\mathbf{c_1} = \langle 0, 3, -2 \rangle$$

Therefore,

$$\mathbf{r}(t) = \left\langle 4\sqrt{t}, 6t, 4t^2 \right\rangle + \langle 0, 3, -2 \rangle = \left\langle 4\sqrt{t}, 6t + 3, 4t^2 - 2 \right\rangle$$

Then the location of this particle at $t = 4$ is:

$$\mathbf{r}(4) = \langle 8, 27, 62 \rangle.$$

Section 13.3, #23. Find a path that traces the ciecle in the plane $y = 10$ with radius 4 and center(2,10,-3) with constant speed 8.

Solution. We start with the following parametrization of the circle:

$$\mathbf{r}(t) = \langle 2, 10, -3\rangle + 4\langle \cos t, 0, \sin t\rangle = \langle 2 + 4\cos t, 0, -3 + 4\sin t\rangle$$

We need to reparametrize the curve by making a substitution $t = g(s)$, so that the new parametrization $\mathbf{r_1}(s) = \mathbf{r}(g(s))$ satisfies $\|\mathbf{r_1'}(s)\| = 8$ for all s. We find $\mathbf{r_1'}(s)$ using the Chain Rule:

$$\mathbf{r_1'}(s) = \frac{d}{ds}\mathbf{r}(g(s)) = g'(s)\mathbf{r}'(g(s)) \tag{1}$$

Next, we differentiate $\mathbf{r}(t)$ and then replace t by $g(s)$:

$$\mathbf{r}'(g(s)) = \langle -4\sin g(s), 0, 4\cos g(s)\rangle$$

Substituting in (1) we get:

$$\mathbf{r_1'}(s) = g'(s)\langle -4\sin g(s), 0, 4\cos g(s)\rangle$$

Hence,

$$\|\mathbf{r_1'}(s)\| = 4|g'(s)|\sqrt{(\sin g(s))^2 + (-\cos g(s))^2} = 4|g'(s)|$$

To satisfy $\|\mathbf{r_1'}(s)\| = 8$ for all s, we choose $g'(s) = 2$. We may take the antiderivative $g(s) = 2s$, and obtain the following parametrization:

$$\mathbf{r_1}(s) = \mathbf{r}(g(s)) = \mathbf{r}(2s) = \langle 2 + 4\cos(2s), 10, -3 + 4\sin(2s)\rangle.$$

This is a parametrization of the given circle, with constant speed 8.

Section Section 13.5, #15. Given $\mathbf{a}(t) = \langle t, 4\rangle$, $\mathbf{v}(0) = \langle 3, -2\rangle$, $\mathbf{r}(0) = \langle 0, 0\rangle$
Find $\mathbf{v}(t)$, $\mathbf{r}(t)$

Solution. We first integrate $\mathbf{a}(t)$ to find the velocity vector:

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2}, 4t \right\rangle$$

Substituting the initial condition:

$$\mathbf{v}(t) = \left\langle \frac{t^2}{2}, 4t \right\rangle + \langle 3, -2\rangle = \left\langle \frac{t^2}{2} + 3, 4t - 2 \right\rangle$$

We now integrate the velocity vector to find $\mathbf{r}(t)$:

$$\mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \left\langle \frac{u^2}{2} + 3, 4u - 2 \right\rangle du = \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t \right\rangle$$

Plug in the initial condition $\mathbf{r}(0)$:

$$\mathbf{r}(t) = \left\langle \frac{t^3}{6} + 3t, 2t^2 - 2t \right\rangle.$$