1 Introduction

In this class so far, you have learnt two formulae for the tangent planes. One is in page 806, section 14.4 of the book, which states that the tangent plane for the surface $z = f(x, y)$ at $(x, y) = (a, b)$ is given by the formula

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The other formula is in page 822, at the end of section 14.5 of the book, which states that the tangent plane for the surface $F(x, y, z) = k$ where k is a constant, at $(x, y, z) = (a, b, c)$ is given by the formula

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

In dot product notation, this can be rewritten as

$$\nabla F(a, b, c) \cdot (x - a, y - b, z - c) = 0.$$

What is the difference between these two formulae? When shall we use which one of them? We are going to looking at these two questions in this review note.

2 Explicit v.s. Implicit Formula

The formula in section 14.4 is an explicit formula; while the formula in section 14.5 is an implicit formula. What do I exactly mean here? If you have a surface $z = f(x, y)$, this is an explicit description of the surface, z is some function depending on x and y. However, if you have $F(x, y, z) = k$, this is an implicit description of the surface, although z is some function depending on x and y, you don't have an explicit formula for z, the variables x, y and z all couple together. For the explicit surface $z = f(x, y)$, we have the formula

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

to describe the tangent plane; while for the implicit surface $F(x, y, z) = k$, we have the formula

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

to describe the tangent plane.

Now, notice that an explicit surface can always be rewritten as an implicit surface. If you have $z = f(x, y)$, you can always let $F(x, y, z) = f(x, y) - z$. Then $F(x, y, z) = 0$ represents the same surface as $z = f(x, y)$. Now if I say that these two surfaces $F(x, y, z) = 0$ and $z = f(x, y)$ have the tangent plane at a given point $(x, y, z) = (a, b, c)$, where $c = f(a, b)$, do you agree with me?

This should be true because $F(x,y,z) = 0$ and $z = f(x,y)$ are the same surface. However, at first, you may have questions about it because we use different formulae for the tangent planes for these two surfaces. So should the tangent planes really be the same even if you calculate it differently? The answer is yes! Let me show you why.

If you have $z = f(x,y)$, then the tangent plane at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

If you have $F(x,y,z) = f(x,y) - z = 0$, the tangent plane at (a,b,c) is given by

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

Notice that $F(x,y,z) = f(x,y) - z$, so $F(a,b,c) = 0$ implies that $f(a,b) - c = 0$, which says $c = f(a,b)$. Since $F(x,y,z) = f(x,y) - z$, we have $F_x(a,b,c) = f_x(a,b)$, $F_y(a,b,c) = f_y(a,b)$, and $F_z(a,b,c) = -1$. This implies that the formula

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

can be rewritten as

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) + (-1)(z - f(a,b)) = 0.$$

This formula is the same as

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Thus we have shown that**the two formulae for the tangent planes are actually the same**. I hope that I have convinced you. But if you are still confused, welcome to discuss with me further.

3 When to Use Which?

If the problem gives you an explicit description of the surface, it's better to use the formula

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

If the problem gives you an implicit description of the surface, it's better to use the formula

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

Example: Given a surface $z = f(x,y) = x^2 + xy + y^2$, find the equation for the tangent plane at $(x,y) = (1,1)$.

Solution: In this case you have an explicit surface, because z is expressed explicitly as $f(x,y)$. So we can apply the formula

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Before we do that, we need to figure out what are a, b, $f(a,b)$, $f_x(a,b)$ and $f_y(a,b)$. Because we are aiming at $(x,y) = (1,1)$, we have $a = 1$ and $b = 1$, hence the function value $f(a,b) = 1^2+1+1^2 = 3$. Now we need to calculate the partial derivatives, $f_x(x,y) = 2x+y$, and $f_y(x,y) = x+2y$, hence $f_x(a,b) = 3$ and $f_y(a,b) = 3$. Finally, applying the formula we get

$$z = 3 + 3(x-1) + 3(y-1).$$

Example: Given a surface $F(x,y,z) = x^2 + xy + y^2 + z^2 = 4$, find the equation for the tangent plane at $(x,y,z) = (1,1,1)$.

Solution: In this case you have an implicit surface, because z is embedded implicitly in the formula $F(x,y,z) = 4$. So we need to use the formula

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

Before we do that, we need to figure out what are $F_x(a,b,c)$, $F_y(a,b,c)$, and $F_z(a,b,c)$. Taking partial derivatives, we get $F_x(x,y,z) = 2x+y$, $F_y(x,y,z) = x+2y$ and $F_z(x,y,z) = 2z$. Plugging in the numbers we get $F_x(a,b,c) = 3$, $F_y(a,b,c) = 3$ and $F_z(a,b,c) = 2$. Finally we apply the formula and get the tangent plane

$$3(x-1) + 3(y-1) + 2(z-1) = 0.$$