1 Review on Single Integrals

Single Integral is the building block of double and triple integrals, so if you are not able to do single integrals, you can not do double and triple integrals. There are three techniques used for single integrals: plain integral; substitution; integration by parts.

1.1 Plain Integral

It is the most straightforward case, you can just apply a formula to get the answer.

1. **Example:**

$$\int_0^1 3s^2 ds.$$

Solution: To solve this, one just apply the formula that $(s^n)' = ns^{n-1}$ and get

$$\int_0^1 3s^2 ds = s^3\Big|_0^1 = 1.$$

2. If you have a constant C, then

$$\int_a^b Cf(s)ds = C\int_a^b f(s)ds, \qquad \text{and} \qquad \int_a^b [C + f(s)]\, ds = (b-a)C + \int_a^b f(s)ds$$

Example:

$$\int_0^2 [5 + 6e^s]\, ds.$$

Solution: It is the same as

$$2 \times 5 + 6\int_0^2 e^s ds = 10 + 6e^s\Big|_0^2 = 10 + 6(e^2 - 1).$$

1.2 Substitution

It is commonly known as "u-subs", although I'm thinking why it is not called "I-subs" or "He-subs", or "She-subs", why it has to be "u-subs"? (*Sorry, just being bored, we call it a cold joke in Chinese.*)

1. **Example:**

$$\int_0^2 2se^{s^2} ds$$

Solution: Make a u-subs, let $u = s^2$, then $u' = 2s$, and $(e^u)' = u'e^u = 2se^s$, hence

$$\int_0^2 2se^{s^2} ds = e^{s^2}\Big|_0^2 = e^4 - 1$$

2. **Examples:**

$$\int_0^{\pi} 2\cos\theta \sin\theta d\theta.$$

Solution: Notice that $\cos\theta' = -\sin\theta$, hence if you let $u(\theta) = \cos\theta$, then $(u^2(\theta))' = 2u(\theta)u'(\theta) = -2\cos\theta\sin\theta$, hence

$$\int_0^{\pi} 2\cos\theta \sin\theta d\theta = -\cos\theta^2\Big|_0^{\pi} = 0.$$

3. As you may have noticed, this kind of integration does require some observation. You need some amount of practices in order to master this technique. But I personally think if you do it in a correct way and be concentrate, you just need a couple of hours and less than ten problems to practice on this, then you will be a master of "u-subs". The key is don't look at the answer, try to figure it out yourself. Here I'm giving you six problems. Try your best to solve them using "u-subs"! Be confident on yourself! If you desire to improve your ability to do integration, but stuck anywhere in these exercises, do email me or post it on Piazza, and I will teach you futher.

(0). $\displaystyle\int_0^1 2s(s^2+1)ds$

(1). $\displaystyle\int_0^2 s^2 e^{s^3} ds$

(2). $\displaystyle\int_0^{\pi} \cos^2\theta \sin\theta d\theta$

(3). $\displaystyle\int_0^{\pi} \sin\theta e^{\cos\theta} d\theta$

(4). $\displaystyle\int_2^3 \frac{\ln x}{x} dx$

(5). $\displaystyle\int_0^1 \frac{\ln(x+2)}{x+2} dx$

1.3 Integration by Parts

Integration by part is a very important technique, many of you have learnt this from 20A or 20B. However, since it is a little bit difficult, and many of you might have already forget about it, I won't test you on problems that you have to use this particular technique. But it is for your benefit if you could learn or review this topic.

1. The formula is

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

for indefinite integral, and

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)dx$$

for definite integral.

2. **Example:**

$$\int_0^1 se^s ds$$

Solution: Let $u(s) = s$, and $v'(s) = e^s$, then $v(s) = e^s$, and $u'(s) = 1$, so

$$\int_0^1 se^s ds = se^s\Big|_0^1 - \int_0^1 e^s ds = se^s\Big|_0^1 - e^s\Big|_0^1 = 1$$

3. **Example:**

$$\int_0^\pi \theta \cos\theta d\theta$$

Solution: Let $u(\theta) = \theta$, $v'(\theta) = \cos\theta$, then $v(\theta) = \sin\theta$ and $u'(\theta) = 1$. So

$$\int_0^\pi \theta\cos\theta d\theta = \theta\sin\theta\Big|_0^\pi - \int_0^\pi \sin\theta d\theta = \theta\sin\theta\Big|_0^\pi + \cos\theta\Big|_0^\pi = \pi\sin\pi + \cos\pi - 1 = -2.$$

4. As you can see, this technique, the integration by parts, also needs some observation, and need some amount of exercise for you to get familiar with.

2 Double and Triple Integrals

This section is about what you are going to be tested on double and triple integrals in the final.

2.1 Double Integrals

1. I will test you on double integral with rectangular domains.

   Example:

$$\int_0^1 \int_1^2 xy\mathrm{d}x\mathrm{d}y$$

Solution: You can regard x and y as two independent variables, that means when you integrate with respect to x, you regard y as a constant; and when you integrate with respect to y, you take x as a costant. Hence, there are two ways to evaluate this integral, either

$$\int_0^1 \int_1^2 xy\mathrm{d}x\mathrm{d}y = \int_0^1 \left(y\int_1^2 x\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \left(y\frac{x^2}{2}\Big|_1^2\right)\mathrm{d}y = \int_0^1 \frac{3y}{2}\mathrm{d}y = \frac{3y^2}{4}\Big|_0^1 = \frac{3}{4},$$

or

$$\int_0^1 \int_1^2 xy\mathrm{d}x\mathrm{d}y = \int_0^1 y\mathrm{d}y \cdot \int_1^2 x\mathrm{d}x = \frac{y^2}{2}\Big|_0^1 \cdot \frac{x^2}{2}\Big|_1^2 = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}.$$

Remark: *The symbols $\cdot$ and $\times$ in the above calculation both represent the usual multiplication, not dot product and cross product.*

2. Sometimes, I will describe the domain as $\mathcal{D} = [a, b] \times [c, d]$, and you need to figure out how to write down the iterated integrals. An example of this is

   Example:

$$\iint_{\mathcal{D}} xy \mathrm{d}A, \qquad \text{with} \qquad \mathcal{D} = [1, 2] \times [0, 1].$$

   Solution: You need to rewrite the integral as an iterated integral, then calculate the integral. How to write the iterated integral? First, x is going from 1 to 2, and our integrand is xy, so we have

$$\int_1^2 xy \mathrm{d}x.$$

   Second, y is going from 0 to 1, so we have

$$\boxed{\int_0^1 \left(\int_1^2 xy \mathrm{d}x \right) \mathrm{d}y.}$$

   Alternatively, you can have y going from 0 to 1 first, so you have

$$\int_0^1 xy \mathrm{d}y,$$

   and then x going from 1 to 2, so you have

$$\boxed{\int_1^2 \left(\int_0^1 xy \mathrm{d}y \right) \mathrm{d}x.}$$

   These two expressions are the same, because of Fubini's theorem. The remaining step is to evaluate the integral. As what we did from the previous example, we have either

$$\int_0^1 \int_1^2 xy \mathrm{d}x \mathrm{d}y = \int_0^1 \left(y \int_1^2 x \mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(y \frac{x^2}{2} \bigg|_1^2 \right) \mathrm{d}y = \int_0^1 \frac{3y}{2} \mathrm{d}y = \frac{3y^2}{4} \bigg|_0^1 = \boxed{\frac{3}{4}},$$

   or

$$\int_0^1 \int_1^2 xy \mathrm{d}x \mathrm{d}y = \int_0^1 y \mathrm{d}y \cdot \int_1^2 x \mathrm{d}x = \frac{y^2}{2} \bigg|_0^1 \cdot \frac{x^2}{2} \bigg|_1^2 = \frac{1}{2} \times \frac{3}{2} = \boxed{\frac{3}{4}}.$$

3. Double integrals over a general domain can also be tested in the final.

   Example:

$$\iint_{\mathcal{D}} xy \mathrm{d}x \mathrm{d}y, \qquad \text{with} \qquad \mathcal{D} = \{(x, y) : 0 \leq x \leq 1 - y, 0 \leq y \leq 1\}$$

   Solution:

   - First, you need to know what the notation $\mathcal{D} = \{(x, y) : 0 \leq x \leq 1 - y, 0 \leq y \leq 1\}$ mean. It means that $\mathcal{D}$ is a set composed with all those points (x, y) with $0 \leq x \leq 1 - y$ and $0 \leq y \leq 1$.

- Next, how do we write down the iterated integral? From $0 \leq x \leq 1-y$, we have

  $$\int_0^{1-y} xy\mathrm{d}x;$$

  from $0 \leq y \leq 1$, we have

  $$\boxed{\int_0^1 \left(\int_0^{1-y} xy\mathrm{d}x\right) \mathrm{d}y}$$

- Next, we evaluate the integral. Notice that y is a constant relative to x, so we can take y out when we integrate over x, this gives us

  $$\int_0^{1-y} xy\mathrm{d}x = y\int_0^{1-y} x\mathrm{d}x = y\frac{x^2}{2}\bigg|_0^{1-y} = \frac{y(1-y)^2}{2}.$$

  This gives us the inner integral, now the outer integral becomes

  $$\int_0^1 \frac{y(1-y)^2}{2}\mathrm{d}y.$$

  To solve it, you need to use complete square formula to expand the term $y(1-y)^2 = y(1-2y+y^2) = y - 2y^2 + y^3$, and then you get

  $$\int_0^1 \frac{y(1-y)^2}{2}\mathrm{d}y = \int_0^1 \frac{y-2y^2+y^3}{2}\mathrm{d}y = \left(\frac{y^2}{4} - \frac{y^3}{3} + \frac{y^4}{8}\right)\bigg|_0^1 = \frac{1}{4} - \frac{1}{3} + 18 = \boxed{\frac{1}{24}}.$$

4. Transform between vertical simple and horizontal simple cases is also required in the final.

   Example:

   $$\iint_{\mathcal{D}} xy\mathrm{d}x\mathrm{d}y, \qquad \text{with} \qquad \mathcal{D} = \{(x,y) : 0 \leq x \leq 1-y, 0 \leq y \leq 1\}$$

   Do the integral again with integrating along y first, then x.

   Solution: In the previous example, we do the integral along x first, which is the horizontal simple case, now we want to do the integral along y first, which is the vertical simple case.

   - First, we change it to vertical simple case. From the condition that $0 \leq x \leq 1-y$ and $0 \leq y \leq 1$, in particular that $x \leq 1-y$, we have $y \leq 1-x$. Now y needs a lower bound, which comes from $0 \leq y$, in combination we have $0 \leq y \leq 1-x$. Now for x, we have $0 \leq x \leq 1-y$. And since $0 \leq y$, we have $0 \leq x \leq 1$. So we obtain

     $$\boxed{0 \leq y \leq 1-x, \qquad 0 \leq x \leq 1}$$

   - Now the iterated integral can be written as

     $$\boxed{\int_0^1 \int_0^{1-x} xy\mathrm{d}y\mathrm{d}x.}$$

- To evaluate this integral, we follow the similar steps as from the previous example, we don't repeat here.

5. Let me show you another example of transforming between vertical and horizontal simple cases.

   Example:

$$\iint_{\mathcal{D}} f(x,y)\mathrm{d}A, \qquad \text{with} \quad \mathcal{D} = \{(x,y) : 0 \le x \le y^2, 0 \le y \le 1\}$$

   We want to write this double integral as an iterated integral in two ways.

   Solution:

   - From $0 \le x \le y^2$ and $0 \le y \le 1$ we can write down the iterated integral for the horizontal simple case:

$$\boxed{\int_0^1 \left(\int_0^{y^2} f(x,y)\mathrm{d}x\mathrm{d}y \right)}$$

   - For the vertical simple case, we have from $x \le y^2$ that $\sqrt{x} \le y$, combined with $y \le 1$ we have $\boxed{\sqrt{x} \le y \le 1}$. For x we have $0 \le x \le y^2$, with $y \le 1$ we have $\boxed{0 \le x \le 1}$ for x. Now the iterated integral for vertical simple case can be written down as

$$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)\mathrm{d}y\mathrm{d}x.$$

2.2 Triple Integrals

1. Evaluate triple integral over a box.

   Example:

$$\int_0^1 \int_0^1 \int_0^1 x\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

   Solution: You can either do it in the following way:

$$\int_0^1 \int_0^1 \left(\int_0^1 x\mathrm{d}x \right) \mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^1 \left(\frac{x^2}{2}\bigg|_0^1 \right) \mathrm{d}y\mathrm{d}z$$
$$= \int_0^1 \int_0^1 \frac{1}{2}\mathrm{d}y\mathrm{d}z = \int_0^1 \left(\frac{y}{2}\bigg|_0^1 \right) \mathrm{d}z = \int_0^1 \frac{1}{2}\mathrm{d}z = \frac{z}{2}\bigg|_0^1 = \frac{1}{2},$$

   or the other way:

$$\int_0^1 \int_0^1 \left(\int_0^1 x\mathrm{d}x \right) \mathrm{d}y\mathrm{d}z = \int_0^1 x\mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = \frac{x^2}{2}\bigg|_0^1 \cdot y\bigg|_0^1 \cdot z\bigg|_0^1 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$

   Remark: *Again the notations $\cdot$ and $\times$ in the above expression denotes unsual multiplication other than dot product and cross product.*

2. Evaluate triple integral over an irregular domain.

   Example:

$$\iiint_{\mathcal{B}} z\mathrm{d}V \qquad \text{with} \qquad \mathcal{B} = \{(x,y,z) : x^2+y^2 \le 1, 0 \le z \le 1\}$$

   Solution:

   - First we need to know what the notation $\mathcal{B} = \{(x,y,z) : x^2+y^2 \le 1, 0 \le z \le 1\}$ mean. It means that $\mathcal{B}$ is the set containing all points (x,y,z) such that $x^2+y^2 \le 1$ and $0 \le z \le 1$.
   - Now we need to write the iterated integral. We already have $0 \le z \le 1$, now we need to derive some bounds for x and y. Since $x^2+y^2 \le 1$, we have $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, and $-1 \le x \le 1$. (Some may have question here: why we don't write $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$? This expression is correct of course. But we don't write it because our purpose is to derive an iterated integral. The expression that $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ indicates we have integral along y before we have integral along x, so the bounds for x can not depend on y. And the most negative value that $-\sqrt{1-y^2}$ can take is -1, and the most positive value that $\sqrt{1-y^2}$ can take is 1, so we get $-1 \le x \le 1$.) So the integral can be written as

$$\boxed{\iiint_{\mathcal{B}} z\mathrm{d}V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 z\mathrm{d}z\mathrm{d}y\mathrm{d}x.}$$

   - Now we calculate the integral. The innermost integral is

$$\int_0^1 z\mathrm{d}z = \frac{z^2}{2}\bigg|_0^1 = \frac{1}{2}.$$

     The middle integral is

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 z\mathrm{d}z\mathrm{d}y = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{2}\mathrm{d}y = \frac{y}{2}\bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \sqrt{1-x^2}.$$

     The outtermost integral becomes

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 z\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\sqrt{1-x^2}\mathrm{d}x.$$

     This final integral is a little bit complicated, if I ever ask you to do this in the final exam, I will provide you with the formula. However, here let us integrate this one out. The trick is to let $x = \sin\theta$. Then $\sqrt{1-x^2} = \sqrt{1-sin^2\theta} = |\cos\theta|$, and $x'(\theta) = \cos\theta$. x integrating from -1 to 1 is the same as θ integrating from $-\pi/2$ to $\pi/2$, because as θ goes from $-\pi/2$ to $\pi/2$, $x = \sin\theta$ goes from -1 to 1. Hence the integral can be rewritten as

$$\int_{-1}^{1}\sqrt{1-x^2}\mathrm{d}x = \int_{-\pi/2}^{\pi/2}\cos\theta\cos\theta\mathrm{d}\theta.$$

Notice that we take away the absolute value symbols for $\cos\theta$, because for θ from $-\pi/2$ to $\pi/2$, $\cos\theta \geq 0$ always. Now we need to use the formula that $\cos(2\theta) = 2\cos^2\theta - 1$, and we get

$$\int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}\mathrm{d}\theta = \left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2} = \boxed{\pi}.$$

This is our final answer to this problem.

3. It is also possible that you will be asked to evaluate the integral over a region which is bounded between two intesecting surfaces.

Example:

$$\iiint_{\mathcal{B}} f(x,y,z)\mathrm{d}V, \qquad \text{with} \qquad f(x,y,z) = 1,$$

and $\mathcal{B}$ is the domain bounded above by $z = 3 - 2x^2 - y^2$ and below by $z = x^2 + 2y^2$.

Solution: The surfaces are plotted in Figure 1. The region $\mathcal{B}$ is simply the region in between these two surfaces. This region can be described as composed of all points (x, y, z), with (x, y) inside $\mathcal{D}$, the brown colored domain in the xy-plane, and $x^2 + 2y^2 \leq z \leq 3 - 2x^2 - y^2$. Here for your understanding, I draw the picture, however, you don't really need this picture in order to solve the problem. Here are the steps.

- First, obtain the expression for the curve $\mathcal{C}$, which is the boundary of $\mathcal{D}$. This can be done via setting

$$3 - 2x^2 - y^2 = x^2 + y^2,$$

which gives us $\boxed{x^2 + y^2 = 1}$. This is the expression of $\mathcal{C}$, and hence the domain $\mathcal{D}$ is composed of (x, y) such that $\boxed{x^2 + y^2 \leq 1}$.

- Step II, we derive the expression for the iterated integral. What we already have is that $x^2 + 2y^2 \leq z \leq 3 - 2x^2 - y^2$, and $x^2 + y^2 \leq 1$, now if we integrate y before we integrate x, we need to have a bound for y as a function of x. From $x^2 + y^2 \leq 1$ we have $y^2 \leq 1 - x^2$, hence $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. And now we need to provide a bound for x, which is independent of y. From $x^2 + y^2 \leq 1$ we have $x^2 \leq 1 - y^2$, hence$-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$. Now the most negative value $-\sqrt{1-y^2}$ can take is -1, and the most positive value $\sqrt{1-y^2}$ can take is 1, so a bound for x is that $-1 \leq x \leq 1$. Hence the iterated integral can be written as

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+2y^2}^{3-2x^2-y^2} f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

- The next step is to do the actual integration. With $f(x, y, z) = 1$, the innermost integral is simply

$$\int_{x^2+2y^2}^{3-2x^2-y^2} f(x,y,z)\mathrm{d}z = 3 - 2x^2 - y^2 - (x^2 + y^2) = 3 - 3(x^2 + y^2).$$

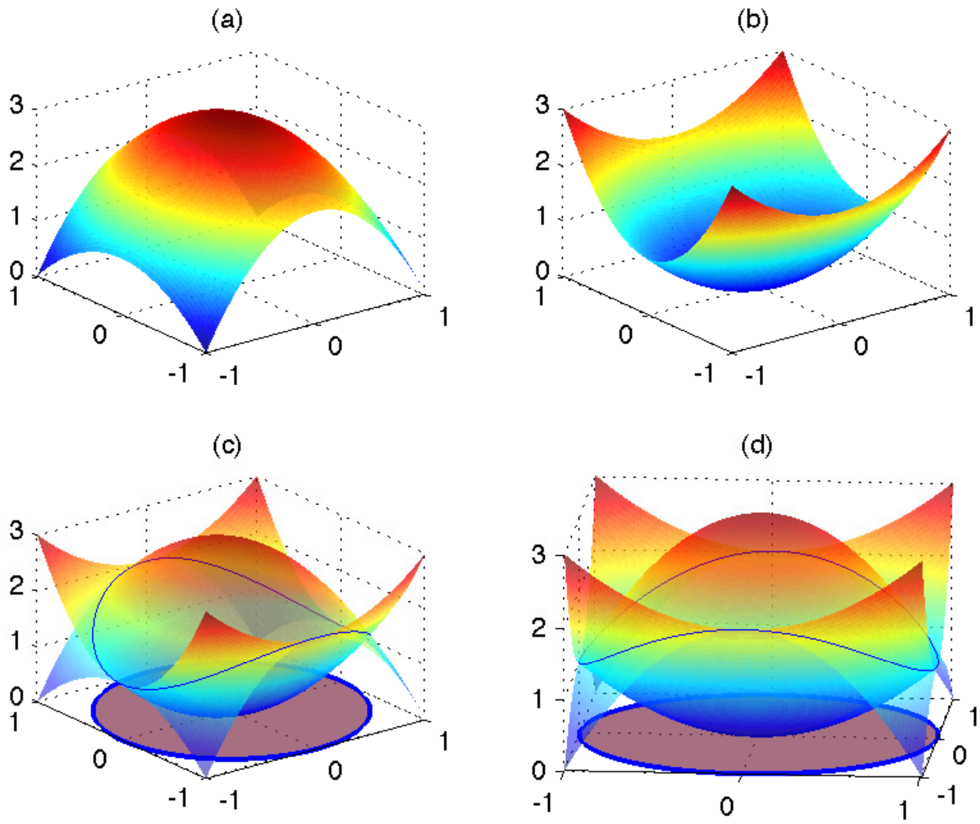

Figure 1: (a): Surface described by the equation $z = 3 - 2x^2 - y^2$; (b): Surface described by the equation $z = x^2 + 2y^2$; (c), (d): The intersection of these two surfaces with the intersection curve, from different angles of view.

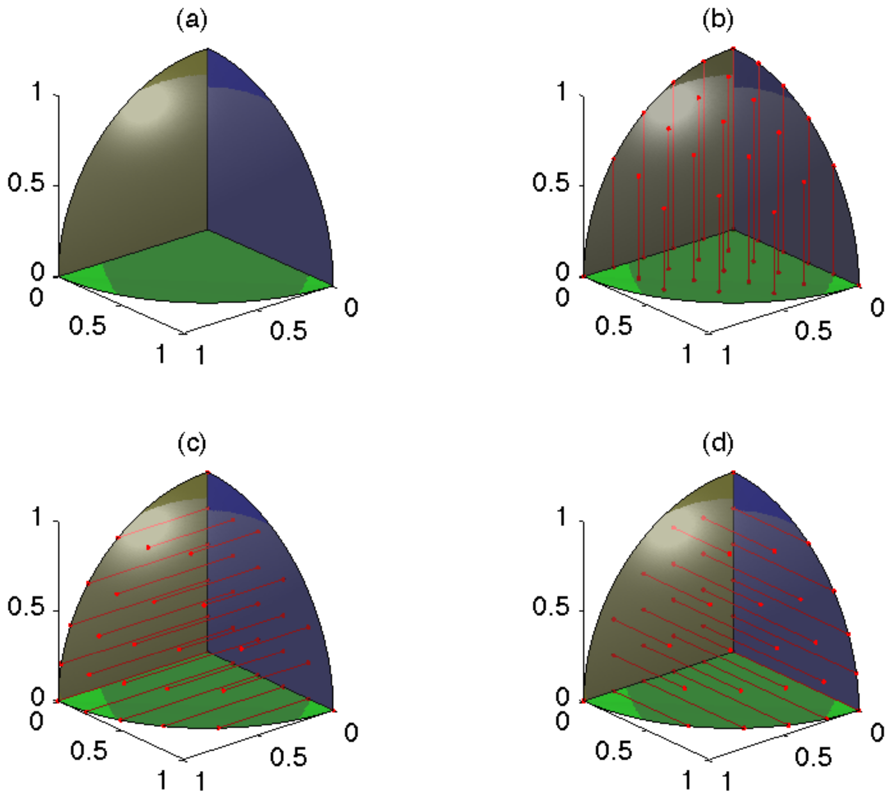

Figure 2: The region $\mathcal{B}$

The next middle integral and the outer integral is complicated, and in fact, if we use polar coordinate, the problem gets much simplified. In the final exams, I won't ask you to compute this kind of very complicated integrals. I may only ask you to derive the expression for the iterated integrals, which is up to the end of step II. If I do ask you to do this kind of complicated integrals, then you need to use polar coordinates to do it. I will come back to this problem when I do review on polar coordinates.

4. To do the actual integrations may be difficult if you are lack of practices. However, I may ask you to do easier stuff that I can test on your understanding of the concept, such as writing down the expression for the iterated integral in different ways.

   Example: Let $\mathcal{B}$ be a region restricted by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2 + y^2 + z^2 \leq 1$. Write down three expressions of iterated integral for

   $$\iiint_{\mathcal{B}} f(x, y, z) \mathrm{d}V$$

   Solution: The region $\mathcal{B}$ is plotted in Figure 2.

- For (b), the lines are vertical, parallel to the $z-$axis. That indicates that we integrate along z first, hence we need to derive a bound for z first. We already have one: $z \geq 0$. We also have $x^2 + y^2 + z^2 \leq 1$, which implies that $z \leq \sqrt{1 - x^2 - y^2}$, hence we have $0 \leq z \leq \sqrt{1 - x^2 - y^2}$. Now we look at the $xy-$plane, what bounds should we give to x and y? Since $z \leq \sqrt{1 - x^2 - y^2}$, we need to have $1 - x^2 - y^2 \geq 0$, which implies that $x^2 + y^2 \leq 1$. Hence we can put $y \leq \sqrt{1 - x^2}$ and $x \leq \sqrt{1 - y^2}$, on the other hand $x \geq 0$ and $y \geq 0$, so $0 \leq y \leq \sqrt{1 - x^2}$ and $0 \leq x \leq \sqrt{1 - y^2}$. Now we have freedom to choose to integrate y before x or x before y. If we choose to integrate y before x, then we have the bound for y already, which depends on x, but we cannot have the bound of x depending on y, so we have to adjust the bound $0 \leq x \leq \sqrt{1 - y^2}$ a little bit. Notice that the most positive value that $\sqrt{1 - y^2}$ can take is 1, so $0 \leq x \leq 1$. Hence the expression of the iterated integral for this case is

$$\boxed{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.}$$

- For (c), the lines are horizontal, parallel to the $x-$axis. That indicates that we integrate along x first, hence we need to derive a bound for x first. We already have one: $x \geq 0$. We also have $x^2 + y^2 + z^2 \leq 1$, which implies that $x \leq \sqrt{1 - z^2 - y^2}$, hence we have $0 \leq x \leq \sqrt{1 - z^2 - y^2}$. Now we look at the $yz-$plane, what bounds should we give to z and y? Since $x \leq \sqrt{1 - z^2 - y^2}$, we need to have $1 - z^2 - y^2 \geq 0$, which implies that $z^2 + y^2 \leq 1$. Hence we can put $y \leq \sqrt{1 - z^2}$ and $z \leq \sqrt{1 - z^2}$, on the other hand $z \geq 0$ and $y \geq 0$, so $0 \leq y \leq \sqrt{1 - z^2}$ and $0 \leq z \leq \sqrt{1 - y^2}$. Now we have freedom to choose to integrate y before z or z before y. If we choose to integrate y before z, then we have the bound for y already, which depends on z, but we cannot have the bound of z depending on y, so we have to adjust the bound $0 \leq z \leq \sqrt{1 - y^2}$ a little bit. Notice that the most positive value that $\sqrt{1 - y^2}$ can take is 1, so $0 \leq z \leq 1$. Hence the expression of the iterated integral for this case is

$$\boxed{\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.}$$

- For (d), the lines are horizontal, parallel to the $y-$axis. That indicates that we integrate along y first. Basically following the same procedure as in the previous two cases, we get a similar answer:

$$\boxed{\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z)\mathrm{d}y\mathrm{d}x\mathrm{d}z.}$$