11/18 $\boxed{\text{Review}}$

* Markov decision process

$MDP = \{ \mathcal{S}, \mathcal{A}, P(s'|s,a), R(s) \}$

* Policy = deterministic mapping $\pi : \mathcal{S} \to \mathcal{A}$
* Value functions

$$V^\pi(s) = E^\pi \left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \,\middle|\, s_0 = s \right]$$

$$Q^\pi(s,a) = E^\pi \left[\sum_{t=0}^{\infty} \gamma^t R(s_t) \,\middle|\, s_0 = s, a_0 = a \right]$$

* Policy evaluation

Solve linear equations : $\sum_{s'} [I(s,s') - \gamma P(s'|s,\pi(s))] V^\pi(s') = R(s)$

* Policy improvement

Greedy policy $\pi'(s) = \underset{a}{\operatorname{argmax}} \; Q^\pi(s,a)$

Thm : $V^{\pi'}(s) \geq V^\pi(s)$ for all states s

* Policy iteration

$$\pi_0 \xrightarrow{\text{evaluate}} \begin{matrix} V^{\pi_0}(s) \\ Q^{\pi_0}(s) \end{matrix} \xrightarrow{\text{improve}} \pi_1 \longrightarrow \cdots$$

Thm : if $\pi'(s) = \underset{a}{\operatorname{argmax}} \; Q^\pi(s,a)$ and $V^{\pi'}(s) = V^\pi(s)$ for all s,

then $V^\pi(s) = V^*(s)$.

Proof : 1) From last time :

$$V^\pi(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) \, V^\pi(s') \quad : \text{Bellman optimality equation.}$$

2) Iterate RHS :

$$V^\pi(s) = R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) \left[R(s') + \gamma \max_{a'} \sum_{s''} P(s''|s',a') \, V^\pi(s'') \right] \quad : \text{B}$$

Imagine iterating above t times.

Let's show that this iterated expression implies optimality.

Let $\hat{\pi}(s)$ be any other policy :

$$V^{\hat{\pi}}(s) = R(s) + \gamma \sum_{s'} P(s'|s,\hat{\pi}(s)) \, V^{\hat{\pi}}(s') \quad : \text{Bellman equation}$$

$$\leq R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) \, V^{\hat{\pi}}(s') \quad : \text{greedy}$$

$$= R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) \left[R(s') + \gamma \sum_{s''} P(s''|s',\hat{\pi}(s')) \, V^{\hat{\pi}}(s'') \right] \quad : \text{iterate}$$

$$V^{\hat{\pi}}(s) \leq R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) \left[R(s') + \gamma \max_{a'} \sum_{s''} P(s''|s',a') \, V^{\hat{\pi}}(s'') \right] \quad : \text{be greedy}$$

$$= \cdots \leq \cdots = \cdots \quad : \text{A}$$

Consider upper bound on $V^{\hat{\pi}}(s)$ by iterating t times. (being greedy then applying Bellman equation.)

Compare to equality after t iterations for $V^\pi(s)$.

As $t \to \infty$, RHS on upper bound on $V^{\hat{\pi}}(s)$ converges to RHS of equality for $V^\pi(s)$.

Thus, as $t \to \infty$: $V^{\tilde{\pi}}(s) \leq \lim_{t\to\infty} [\text{Ⓐ}] = \lim_{t\to\infty} [\text{Ⓑ}] = V^{\pi}(s)$

Thus, for all policies $\tilde{\pi}(s)$ and states s: $V^{\pi}(s) \geq V^{\tilde{\pi}}(s)$

$$V^{\pi}(s) = \max_{\tilde{\pi}} V^{\tilde{\pi}}(s) \longrightarrow V^{\pi}(s) = V^{*}(s).$$

Finally, once you have $V^{*}(s)$,

$$\pi^{*}(s) = \arg\max_{a} Q^{*}(s,a)$$
$$= \arg\max_{a} \sum_{s'} P(s'|s,a)\, V^{*}(s')$$

* Pros & Cons of policy iteration.

(+) converges quickly in finite # steps

(−) requires policy evaluation $O(n^3)$ at each iteration.

$\boxed{\text{Value iteration}}$

* How to compute $V^{*}(s)$ directly?
* Bellman optimality equation.

$$V^{*}(s) = \max_{a} Q^{*}(s,a)$$
$$= \max_{a} \left[R(s) + \gamma \sum_{s'} P(s'|s,a)\, V^{*}(s') \right]$$
$$= R(s) + \gamma \max_{a} \sum_{s'} P(s'|s,a)\, V^{*}(s') \qquad (*)$$

Before we showed that if $V^{\pi}(s)$ satisfied $(*)$, then π is optimal.

Here, we show that $V^{*}(s)$ must satisfy $(*)$.

Q: How to solve this set of N nonlinear equations for N unknowns?

* Algorithm

1) initialize: $V_0(s) = 0$ for all s.

estimate at k^{th} iteration

2) iterate: $V_{k+1}(s) = R(s) + \gamma \max_{a} \left[\sum_{s'} P(s'|s,a)\, V_k(s') \right]$ for all $s = 1, 2, \cdots, n$

Note: this algorithm works on value functions, not policies.

But, incremental policies can be computed as:

$$\pi_{k+1}(s) = \text{greedy}[V_k(s)] = \arg\max_{a} \left[\sum_{s'} P(s'|s,a)\, V_k(s') \right]$$

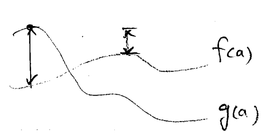
3) after convergence:

$$\pi^{*}(s) = \arg\max_{a} \left[Q^{*}(s,a) \right] = \arg\max_{a} \left[\sum_{s'} P(s'|s,a)\, V^{*}(s') \right]$$

Does algorithm converge? $V^{*}(s)$ is obviously a fixed point.

But are there others? Does it always reach $V^{*}(s)$?

* Lemma: $\left| \max_{a}[f(a)] - \max_{a}[g(a)] \right| \leq \max_{a} |f(a) - g(a)|$



Proof of lemma:

For all a: $f(a) - \max_{a'} g(a') \leq f(a) - g(a)$

Max over a: $\max_{a} f(a) - \max_{a'} g(a') \leq \max_{a} [f(a) - g(a)] \leq \max_{a} |f(a) - g(a)|$

By symmetry: $\max_{a} g(a) - \max_{a'} f(a') \leq \max_{a} |f(a) - g(a)|$.

Combining last two bounds proves lemma.

* Theorem : Value iteration converges.

$$\lim_{k\to\infty} [V_k(s)] \longrightarrow V^*(s) \quad \text{for all states } s.$$

* Proof : let $\Delta_k = \max_s |V_k(s) - V^*(s)|$ error at k^{th} iteration.

$$\Delta_{k+1} = \max_s |V_{k+1}(s) - V^*(s)|$$

$$= \max_s \Big| [R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V_k(s')] \quad \text{from value iteration}$$

$$- [R(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^*(s')] \Big| \quad \text{from Bellman optimality}$$

$$= \gamma \max_s \Big| \max_a \underbrace{\sum_{s'} P(s'|s,a) V_k(s')}_{f(a)} - \max_a \underbrace{\sum_{s'} P(s'|s,a) V^*(s')}_{g(a)} \Big| \quad \text{Apply lemma}$$

$$\Delta_{k+1} \le \gamma \max_s \max_a \Big| \sum_{s'} P(s'|s,a) V_k(s') - \sum_{s'} P(s'|s,a) V^*(s') \Big|$$

$$= \gamma \max_s \max_a \Big| \sum_{s'} P(s'|s,a) [V_k(s') - V^*(s')] \Big|$$

$$\le \gamma \max_s \max_a \Big| \sum_{s'} P(s'|s,a) \max_{s''} |V_k(s'') - V^*(s'')| \Big|$$

$$= \gamma \max_s \max_a \Big| \sum_{s'} P(s'|s,a) \Delta_k \Big|$$

$$= \gamma \Delta_k \max_s \max_a (1)$$

Hence : $\Delta_{k+1} \le \gamma \Delta_k$ with $\gamma < 1$. so-called "contraction" mapping.

By iteration : $\Delta_k \le \gamma^k \Delta_0 \longrightarrow 0$ as $k \to \infty$.

Assume rewards are bounded : $\Delta_0 = \max_s |V_0(s) - V^*(s)|$

$$= \max_s |V^*(s)|$$

$$\le \max_s |R(s)| (1 + \gamma + \gamma^2 + \cdots)$$

$$= \max_s |R(s)| \cdot \left(\frac{1}{1-\gamma}\right)$$

More iterations required as $\gamma \to 1$.