TOPOLOGY PRESENTATION

DAVID SIDI

We want to know how the injectivity of a composition relates to the injectivity of the 'outer' function in the composition.

Problem. For $f : A \to B$ and $g : B \to C$, show that$(g \circ f) : A \to C$ may be injective while g is not injective.

Example. Suppose $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{1, 2\}$. Define $g(1) = 1$, $g(2) = 2$, $g(3) = 2$, and define $f(1) = 1$, $f(2) = 2$. $g \circ f$ is the identity on A, so since no distinct elements of A share a value under the identity in $C = A$, $g \circ f$ is injective. However, since $g(2) = g(3)$, g is not injective. See figure 0.1.

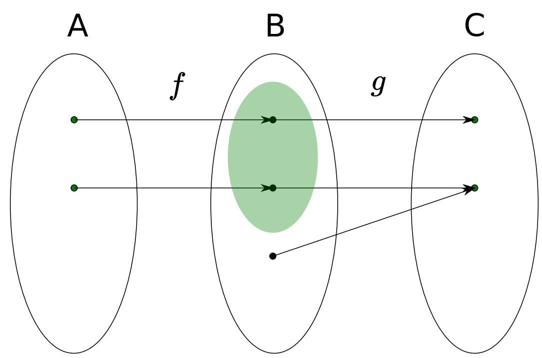

FIGURE 0.1. Mapping diagram for the example given in the example given for problem 1. Note that $f(A)$ is represented as a green region.

Problem. For $f : A \to B$ and $g : B \to C$, show that $(g \circ f) : A \to C$ is injective implies that $g \restriction_{f(A)}$ is also injective.

Proof. Let $\{b_1, b_2\} \subset f(A)$, and suppose $g \restriction_{f(A)} (b_1) = g \restriction_{f(A)} (b_2)$: then for some $\{a_1, a_2\} \subset A$, $f(a_1) = b_1$ and $f(a_2) = b_2$. Thus

$$g \restriction_{f(A)} (b_1) = g \restriction_{f(A)} (f(a_1))$$

and

$$g \restriction_{f(A)} (b_2) = g \restriction_{f(A)} (f(a_2)).$$

But by the definition of restriction, $g \restriction_{f(A)} (f(a_1)) = g(f(a_1))$, which is $(g \circ f)(a_1)$,[1] and similarly for a_2. Thus by injectivity of $(g \circ f)$, we have $a_1 = a_2$, and so $b_1 = b_2$. $\square$

It is straightforward to show the remaining parts of the following (very simple) theorem.

Theorem. *$g \circ f$ is injective iff f and $g \restriction_{f(A)}$ are both injective.*

[1] It can be shown that $(g \circ f)(x) = g(f(x))$ for x in the domain of f.