Basi(c)s of a Topology

To check if something is a topology:

check the definition:

Definition. A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

To check if something is a basis for a topology:

check the definition:

Definition. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If $x$belongs to the intersection of two basis elements B_1 and B_2, there there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

To check if something is a subbasis for a topology:

check the definition:

Definition. If X is a set, a **subbasis** for a topology on X is a collection $\mathcal{B}$ of subsets of X whose union equals X.

The topology generated by the subbasis is defined to be the collection of all unions of finite intersections of elements of the subbasis.

To find the topology generated by a basis:

check the definition:

We define the topology generated by $\mathcal{B}$ as follows: A subset U of X is open in X if for each $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

use Lemma 13.1

Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

To check if something is a basis for <u>THE</u> topology:

use Lemma 13.2

Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an elements $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.

Note: here we said that$X$ was a topological space, so we have a topology in mind. We are just then checking if a set we have is a basis for that topology. This lemma gives us both that $\mathcal{C}$ is a basis and generates our known topology.

Use the steps above - check it is a basis and then check it generates the right topology.

To compare topologies:

Use Lemma 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.