1. A map $f : X \to Y$ is said to be an **open map** is for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ is an open map.

   Solution: Let $U \times V$ be an open set of $X \times Y$. I will show that $\pi_1(U \times V) = U$ is open in X. If V is empty, then $U \times V$ is empty and the image of the empty set is the empty set, so that U is open in X. If V is non-empty, then let $y \in V$. For every $x \in U$, $(x, y) \in U \times V$, and so there exists a basis element of $X \times Y$, call it $B_1 \times B_2$, such that $(x, y) \in B_1 \times B_2 \subseteq U \times V$. To see that $B_1 \subseteq U$, we see that if $a \in B_1$, then $(a, y) \in B_1 \times B_2 \subseteq U \times V$, so that $a \in U$. By definition of basis elements, B_1 is an open subset of X. Thus for every $x \in U$, there exists an open set B_1 of X such that $x \in B_1 \subseteq U$. By Homework 3 Problem 8, we have that U is open in X, so that π_1 is indeed an open map. $\blacksquare$

2. Show that the countable collection

   $$\{(a, b) \times (c, d) | a < b, \text{ and } c < d, a, b, c, d \in \mathbb{Q}\}$$

   is a basis for $\mathbb{R}^2$ with the standard topology.

   Solution: Let $\mathcal{B} = \{(a, b) \times (c, d) | a < b, \text{ and } c < d, a, b, c, d \in \mathbb{Q}\}$. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}^2$, with basis $\mathcal{B}' = \{(a, b) \times (c, d) | a < b, \text{ and } c < d, a, b, c, d \in \mathbb{R}\}$. By Theorem 15.1, since the set $\{(a, b) | a < b, a, b \in \mathbb{R}\}$, is a basis for the standard topology on $\mathbb{R}$, we know that indeed $\mathcal{B}'$ is a basis for the standard topology on $\mathbb{R}^2$. Let $\mathcal{T}_{\mathcal{B}}$ be the topology generated by $\mathcal{B}$. Let $x \times y \in \mathbb{R}^2$. Let $B \in \mathcal{B}$ such that $x \times y \in B$. Clearly $B \in \mathcal{B}'$ as well, so that $x \times y \in B \subseteq B$. By Lemma 13.3, $\mathcal{T} \supseteq \mathcal{T}_{\mathcal{B}}$. Let $x' \times y' \in \mathbb{R}^2$ and $B' \in \mathcal{B}'$ such that $x' \times y' \in B'$. We know that $B' = (a', b') \times (c', d')$ where $a', b', c', d' \in \mathbb{R}$. By density or rationals in $\mathbb{R}$, there exists $a, b, c, d \in \mathbb{Q}$ such that $a \in (a', x')$, $b \in (x', b')$, $c \in (c', y')$, and $d \in (y', d')$. By construction, $x' \times y' \in (a, b) \times (c, d) \subseteq (a', b') \times (c', d')$, and since $(a, b) \times (c, d) \in \mathcal{B}$, we have that $\mathcal{T}_{\mathcal{B}} \supseteq \mathcal{T}$. Thus $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ and indeed $\mathcal{B}$ is a basis for $\mathcal{T}$. $\blacksquare$

3. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$. In each case it is a familiar topology.

   Solution: A line L in the plane has the form of $(x, y) \in \mathbb{R}^2$. Points on non-vertical lines are uniquely determined by their x coordinate, whereas points on vertical lines are uniquely determined by their y coordinates.

   First, consider a line L as a subspace of $\mathbb{R}_l \times \mathbb{R}$. A basis for open sets in $\mathbb{R}_l \times \mathbb{R}$ are open sets of the form $[a, b) \times (c, d)$ where $a, b, c, d \in \mathbb{R}$ and $a < b$, $c < d$. To find the open sets of L, we have to consider the non-empty intersections of L with these open sets. First we will consider non-vertical lines. Let

   $$x_1 = \inf\{x : (x, y) \in L \cap [a, b) \times (c, d)\}$$

   $$x_2 = \sup\{x : (x, y) \in L \cap [a, b) \times (c, d)\}.$$

   We will call the corresponding y-values, y_1 and y_2.

   Case one: L first intersects the open rectangle on the left hand side i.e. $x_1 = a$. In this case, the point has the form (a, y_1) where $c < y_1 < d$. This point is in the open rectangle above. To leave the rectangle, we will either have $x > b$ or $y < c$ or $y > d$. Because of the strict inequalities, $x_2 \notin$ the rectangle . Thus the open sets are of the form $[x_1, x_2) \in L$.

   Case two: L first intersects the open rectangle on the bottom or top side, i.e. $y_1 = c$ or $y_1 = d$. This point is not in the open rectangle above. Again, to leave the rectangle either will have $x > b$ or $y > d$

or $x < c$. Because of the strict inequalities, $x_2 \notin$ the rectangle. Thus the open sets are of the form $(x_1, x_2) \in L$.

Thus on any non-vertical lines, the open sets are those that would be open in the lower limit topology, and thus we have induced the lower limit topology.

Now for vertical lines: If L has the form $x = k$, then $L \cap [a, b) \times (c, d)$ will either be empty or will be points of the from (k, y) where $c < y < d$. This means open sets on vertical lines are the same as those corresponding to open sets in $\mathbb{R}$ with the standard topology.

Now, consider a line L as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$ A basis for open sets in $\mathbb{R}_l \times \mathbb{R}_l$ are open sets of the form $[a, b) \times [c, d)$ where $a, b, c, d \in \mathbb{R}$ and $a < b$, $c < d$

First, consider lines with positive slope, horizontal and vertical lines.

Case one: L first intersects the open rectangle on the left hand side i.e. $x_1 = a$. This case applies to all horizontal lines and some with positive slope. In this case, the point has the form (a, y_1) where $c < y_1 < d$. This point is in the open rectangle above. To leave the rectangle, we will either have $x > b$ or $y > d$. (We can't have $y < c$ because of the positive slope). Because of the strict inequalities, $x_2 \notin$ the rectangle . Thus the open sets are of the form $[x_1, x_2) \in L$.

Case two: L first intersects the open rectangle on the bottom side, i.e. $y = c$. Because of positive slope, we cannot first intersect at top. This case applies to all vertical lines and the rest with positive slope. This point is in the open rectangle above. Again, to leave the rectangle either will have $x > b$ or $y > d$. Because of the strict inequalities, $x_2 \notin$ the rectangle. Thus the open sets are of the form $[x_1, x_2) \in L$.

This for all lines with non-negative (possibly infinite) slope, we have that the induced topology in the lower limit topology.

Finally, let L be a line with negative slope. Let $(x, y) \in L$ and let $R = [x, x+1) \times [y, y+1)$ which is open in $\mathbb{R}_l \times \mathbb{R}_l$. Then $L \cap R = \{(x, y)\}$. This is true for arbitrary $|\{(x, y)\} \in L$, thus L has the discrete topology.

4. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Under what conditions does the equality hold?
   Solution:
   Lemma: $[a, b]$ is a closed set containing (a, b).
   Proof of Lemma: $X - [a, b] = (-\infty, a) \cup (b, \infty)$, which is open since it is the union of two open sets and thus $[a, b]$ is closed. Now, let $x \in (a, b)$ then $a < x < b$ so clearly $a \leq x \leq b$ so x is in $[a, b]$. Thus $[a, b]$ is a closed set containing (a, b).

   Now, to show $\overline{(a, b)} \subseteq [a, b]$, recall that $\overline{(a, b)}$ is the intersection of all closed sets containing (a, b). Then let $x \in \overline{(a, b)}$ then x is in every closed set containing (a, b) and $[a, b]$ is a closed set containing (a, b) so $x \in [a, b]$ as needed.

   Let $a_+ = \inf\{x : x > a\}$ and let $b_- = \sup\{x : x < b\}$. Consider the closed set $[x, y]$. If $[x, y]$ contains (a, b), then $x \leq a_+$ and $y \geq b_-$. Thus there exists a closed set containing (a, b) that does not contain a or b if and only if $a_+ \neq a$ or $b_- \neq b$. This is true exactly when a has an immediate successor or b has an immediate predecessor. For example, in the order topology on $\mathbb{Z}$, $\overline{(1, 4)} = [2, 3] \subsetneq [1, 4]$.

5. Let A_α be a subset of a space X. Decide if $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$. If not, is one a subset of the other?

   Solution: Let A_α be a subset of a space X for every index α. I claim that $\overline{\cap A_\alpha} \subseteq \cap \overline{A_\alpha}$. Let $x \in \overline{\cap A_\alpha}$. Let U be an open set containing x. By Theorem 17.5, U must intersect $\cap A_\alpha$ at some point y, so that $y \in (\cap A_\alpha) \cap U$. Since $y \in \cap A_\alpha$, $y \in A_\alpha$ for every index α. Thus for every open set U containing x, $U \cap A_\alpha$ is non-empty for every index α. By Theorem 17.5 we have that $x \in \overline{A_\alpha}$ for every α. Thus $x \in \cap \overline{A_\alpha}$.

The converse, however, is not true. Let $X = \mathbb{R}$. Let $A = (-1, 0)$ and $B = (0, 1)$ be intervals. Then $\overline{A} = [-1, 0]$ and $\overline{B} = [0, 1]$, so that $\overline{A} \cap \overline{B} = \{0\}$. However, $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, since $\emptyset$ is closed. Clearly, $\{0\}$ is not a subset of $\emptyset$, and thus it is not necessarily true that the intersection of closures is a subset of the closure of the intersection. ■

6. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

   Solution: I claim that in finite complement topology, the sequence $x_n = 1/n$ converges to every number in $\mathbb{R}$. Let $r \in \mathbb{R}$. Let $U \subseteq \mathbb{R}$ be an open set containing r. Consider the set $S = \{n \in \mathbb{N} | 1/n \in \mathbb{R} \setminus U\}$. If S is empty, let $N = 1$, and we have that for all $n \geq N$, $1/n \in U$, so that x_n converges to r. If S is non-empty, S must be finite since it is a subset of the complement of U. Let m be the maximum of S, and set $N = m + 1$. Clearly, if $n \geq N$, then $1/n \notin S$ by choice of N, and thus $1/n$ must be in U. Thus x_n converges to r. Either way, x_n converges to r, and we have that x_n converges to every real number. ■

7. Show that the T_1 axiom is equivalent to the condition for each pair of points of X, each has a neighborhood not containing the other.

   Solution: Let X be a topological space. Suppose that X satisfies the T_1 axiom. Let x, y be two distinct points in X. Then we know that $\{x\}$ is closed and $\{y\}$ since finite point sets are closed. But then $X \setminus \{x\}$ is open and contains y, and $X \setminus \{y\}$ is open and contains x. Clearly these are two neighborhoods of x, y such that neither neighborhood contains the other point.

   Now suppose that for each distinct $x, y \in X$, each has a neighborhood not containing the other point. It suffices to show that single point sets are closed in order to show that finite point sets are closed, since the finite union of closed sets is closed. Let $x \in X$. Consider $X \setminus \{x\}$. Let $y \in X \setminus \{x\}$. By assumption, there is a neighborhood U of y that does not contain x. Then $y \in U \subseteq X \setminus \{x\}$. Thus by Homework 3 Problem 8, $X \setminus \{x\}$ is open. Thus $\{x\}$ is closed, as desired. ■

8. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\epsilon - \delta$ definition of continuity implies the open set definition.

   Solution: Let $f : \mathbb{R} \to \mathbb{R}$ such that f satisfies the $\epsilon - \delta$ definition of continuity. Let U be an open set of $\mathbb{R}$. I will show that $f^{-1}(U)$ is open in $\mathbb{R}$. Let $x \in f^{-1}(U)$. By definition, $f(x) \in U$. Since U is open, there exists a basis element (a, b) of $\mathbb{R}$ such that $f(x) \in (a, b) \subseteq U$. Let $\epsilon = \min\{f(x) - a, b - f(x)\}$. Consider the interval $(f(x)-\epsilon, f(x)+\epsilon)$. To see that $(f(x)-\epsilon, f(x)+\epsilon) \subseteq (a, b)$, notice that if $\epsilon = f(x)-a$, then $(f(x)-\epsilon, f(x)+\epsilon) = (a, 2f(x)-a) \subseteq (a, b)$ since $2f(x)-a \leq b$ because $f(x)-a \leq b-f(x)$ by our choice of ϵ. Similarly, if $\epsilon = b-f(x)$, then $(f(x)-\epsilon, f(x)+\epsilon) = (2f(x)-b, b) \subseteq (a, b)$ since $2f(x)-b \geq a$ because $b-f(x) \leq f(x)-a$ by our choice of epsilon. So indeed $(f(x)-\epsilon, f(x)+\epsilon) \subseteq (a, b)$. By the $\epsilon-\delta$ criterion for continuity, there exists $\delta > 0$ such that if $z \in (x-\delta, x+\delta)$, then $f(z) \in (f(x)-\epsilon, f(x)+\epsilon)$. Since $(f(x) - \epsilon, f(x) + \epsilon) \subseteq (a, b) \subseteq U$, this means that for all $z \in (x - \delta, x + \delta)$, $f(z) \in U$, so that by definition $(x - \delta, x + \delta) \subseteq f^{-1}(U)$. Thus we have found a basis element containing x that is contained in $f^{-1}(U)$. Since this is true for arbitrary $x \in f^{-1}(U)$, we have shown that $f^{-1}(U)$ is open. ■

9. Suppose $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

   Solution:
   First attempt: Assume $f : X \to Y$ is continuous and x is a limit point of the subset A of X. Let V be a neighborhood of $f(x)$. Then since f is continuous, $U = f^{-1}(V)$ is a neighborhood of x. Thus there exists $y \in U \cap A$ with $y \neq x$. Since $y \in U$, $f(y) \in f(U) \in V$ and since $y \in A$, $f(y) \in f(A)$. But! $f(y)$ might not be different than $f(x)$. This allows us to create a counterexample:

   Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1$. Then 0 is a limit point of $A = (0, 1)$ but $f(0) = 1$ is not a limit point of $f(A) = \{1\}$, since every neighborhood of 1 only intersects $\{1\}$ in a single point, 1 itself.

10. Let $F : X \times Y \to Z$. We say that F is **continuous in each variable separately** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.
Solution: If F is continuous, the the restriction of F to any subspace is continuous. So $G = F|_{X\times\{y_0\}}$ is continuous. Let V be an open set in Z then $G^{-1}(Z)$ is open in $X \times \{y_o\}$ which means $G^{-1}(Z)$ can be written as $U \times \{y_0\}$ where U is open in X. Then we have that $h^{-1}(V) = U$. And thus h is continuous. The proof for k is analogous.

11. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0 \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous (using techniques of this class).

Solution:

(a) To show this, we will show that the topological definition of continuity if equivalent to the $\delta - \epsilon$ definition, and then just use techniques of calculus to show this function is continuous. We showed that the $\delta - \epsilon$ definition implies our definition, so now we just show the converse. Let $f : X \to Y$ be a continuous function, where $X = Y = \mathbb{R}$. For $x \in X$, and $\epsilon > 0$, consider the open set $V = (f(x) - \epsilon, f(x) + \epsilon)$ in Y. Since f in continuous, $U = f^{-1}(V)$ is an open set in X that contains x. Therefore there exists a basis element (a, b) such that $x \in (a, b) \subseteq U$. Let $\delta = \min(x-a, b-x)$, then $x \in (x - \delta, x + \delta) \subseteq (a, b) \subseteq (U)$ so $f(x - \delta, x + \delta) \subseteq f(U) \subseteq V = (f(x) - \epsilon, f(x) + \epsilon)$, satisfying the $\delta - \epsilon$ definition of continuity.
Now, consider $h(x) = F(x, y_0)$. If $y_0 = 0$, the functions looks like

$$h(x) = F(x \times 0) = \begin{cases} 0/(x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to see that this is the same as $h(x) = 0$ which is continuous. When $y_0 \neq 0$,

$$h(x) = F(x \times y_0) = xy_0/(x^2 + y_0^2).$$

We know using techniques from calculus that this is a continuous functions since it is the quotient of continuous functions where the denominator is nonzero.
Since $F(x, y)$ is symmetric in x and y, we will get that $k(y) = F(x_0, y)$ is also continuous.

(b) The function

$$\begin{aligned} g(x) = F(x \times x) &= \begin{cases} x^2/(x^2 + x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \\ &= \begin{cases} 1/2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{aligned}$$

which is clearly not continuous.

(c) If F were continuous, then it restriction to the subset $\{(x, y) \in \mathbb{R}^2 : x = y\}$ would be continuous, but this would mean g is continuous, which it clearly isn't. Therefore, F must not be continuous.

12. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2 \dots)$ such that $x_i \neq 0$ for only finitely many of i. values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution: Let $\mathbb{R}^\infty = \{(x_n) \in \mathbb{R}^\omega | x_i \neq 0 \text{ for only finitely many } i\}$. When considering $\mathbb{R}^\omega$ in the product topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, that is the closure of $\mathbb{R}^\infty$ is the entire space $\mathbb{R}^\omega$. To see this, we just need to show that $\mathbb{R}^\omega \subseteq \overline{\mathbb{R}^\infty}$. Suppose $(y_n) \in \mathbb{R}^\omega$. Let B be a basis element of $\mathbb{R}^\omega$ under the product topology such that $(y_n) \in B$. By definition of product topology, $B = \prod_{n\in\mathbb{N}} U_n$ where $U_n \neq \mathbb{R}$ for only finitely many n. Thus we can construct a sequence (x_n) where $x_n = 0$ if $U_n = \mathbb{R}$ and x_n is an element of U_n if $U_n \neq \mathbb{R}$. By construction, if $x_n \neq 0$, then $U_n \neq \mathbb{R}$. Thus the set of all n such that $x_n \neq 0$ is a subset of the $\{m \in \mathbb{N} | U_m \neq \mathbb{R}\}$, where the latter set is finite. Since subsets of finite sets are finite, this means that $x_n \neq 0$ for only finitely many n. So $(x_n) \in B \cap \mathbb{R}^\infty$, and thus we have shown that every basis element containing (y_n) intersects $\mathbb{R}^\infty$. By Theorem 17.5, $(y_n) \in \overline{\mathbb{R}^\infty}$, and thus indeed we have that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Now when considering $\mathbb{R}^\omega$ in the box topology, I claim that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. It suffices to show that $\mathbb{R}^\infty$ is closed. Consider the complement $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Let $(x_n) \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Then (x_n) is a sequence such that $x_n \neq 0$ for infinitely many n. Define $S = \{n \in \mathbb{N} | x_n \neq 0\}$. And for each $n \in S$, define $S_n = (0, x_n + 1)$ if $x_n > 0$ and $S_n = (x_n - 1, 0)$ if $x_n < 0$. Now for each $n \in \mathbb{N}$, define $U_n = S_n$ if $n \in S$, and $U_n = \mathbb{R}$ if $n \notin S$. Clearly, $U = \prod_{n\in\mathbb{N}} U_n$ is a basis element of $\mathbb{R}^\omega$ under the box topology because each U_n is open. By construction, $x_n \in U_n$ for all $n \in S$ because $x_n \in (x_n - 1, 0)$ if $x_n < 0$ and $x_n \in (0, x_n + 1)$ if $x_n > 0$. Also, $x_n \in U_n$ for all $n \notin S$ because $U_n = \mathbb{R}$ for $n \notin S$. Thus $(x_n) \in U$. Now to see that $U \subseteq \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, consider an element $(y_n) \in U$. For each $n \in S$, $y_n \neq 0$ because $y_n \in U_n = S_n$. Since S is infinite, there are infinitely many n such that $y_n \neq 0$, and thus (y_n) cannot be in $\mathbb{R}^\infty$. Thus indeed $(y_n) \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, so that $U \subseteq \mathbb{R}^\omega \setminus \mathbb{R}^\infty$ and $(x_n) \in U$. Thus for every element of $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$, we can find a basis element containing that element which is contained in $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$. By definition $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$ is open, and thus $\mathbb{R}^\infty$ is closed. So $\mathbb{R}^\infty = \overline{\mathbb{R}^\infty}$. ■

13. Given sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots).$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology.

Solution: It is easy to see that h is a bijection. The function h^{-1} is defined by

$$h((y_1, y_2, \dots)) = ((y_1 - b_1)/a_1, (y_2 - b_2)/a_2, \dots).$$

To show that h and h^{-1} are both continuous, we can show that open sets in a basis get mapped to open sets for both. Let U be an open set in $\mathbb{R}^\omega$ of the form $U = \prod U_i$, where finitely many $U_i = (c_i, d_i)$ and the rest are $\mathbb{R}$. Let $V_i = (a_i(c_i) + b_i, a_i(d_i) + b_i$ if $U_i \neq \mathbb{R}$ and $V_i = \mathbb{R}$ otherwise. Note that V_i is open and only finitely many will not be $\mathbb{R}$. If given x if $x_i \in U_i$ then $h(x)_i \in V_i$. So $x \in \prod V_i$. Also if $y \in V_i$, $h^{-1}(y) \in U_i$, (when viewed as either the pre-image or the inverse). Thus $h(U) = V$ and $h^{-1}(V) = U$ and therefore h is an homeomorphism in the product topology. If we remove the property that only finitely many U_i are not $\mathbb{R}$, everything still works, thus this is also a homeomorphism in the box topology.