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A real number x is said to be algebraic (over the rationals) if it satisfies some polynomial equation of positive degree:

$$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0. \text{ (*)}$$

with $a_i \in \mathbb{Q}\ \forall i \in \{1, 2, ..., n\}$. Assuming each polynomial equation has only finitely man roots, show the set of algebraic numbers is countable.

Proof. Let X be the set of algebraic numbers over Q. Fix n $\in \mathbb{Z}^+$

Let P_n be the set of polynomials of degree less than or equal to n.

Let I = $\{p \in P_n \mid$ p is of the form (*)$\}$

Let $J_p = \{x \in \mathbb{R} |$ x is a zero of p$\}$. By assumption J_p is finite for all p $\in$ I.

Let $X_n = \bigcup_{p \in I} J_p$.

Then X_n is the set of algebraic numbers under a fixed degree n.

To show X_n is countable it suffices to show I is countable, because then X_n will be a countable union of countable sets.

Let f: I $\to \{1\} \times \prod_{i=1}^n \mathbb{Q}$ by

$\forall\ p = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$

$$f(x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0) = (1,\ a_{n-1},\ ...\ ,\ a_0)$$

It is easy to check that f is injective. But $\{1\} \times \prod_{i=1}^n \mathbb{Q}$ is a finite product of countable sets, so it is countable and thus there exists an injective function g : $\{1\} \times \prod_{i=1}^n \mathbb{Q} \to \mathbb{Z}^+$.

Then $g \circ f$: I $\to \mathbb{Z}^+$ is injective. This proves I is countable therefore proving X_n is countable.

Now X = $\bigcup_{n \in \mathbb{Z}^+} X_n$. So X is a countable union of countable sets and therefore is countable. □