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Theorem: Let $f\colon X \to Y$; let Y be compact Hausdorff.
Then f is continuous if and only if the **graph** of f

$$G_f = \{x \times f(x) \mid x \in X\}$$

is closed in $X \times Y$.
Lemma: If Y is compact then the projection $\pi_1\colon X \times Y \to X$ is a closed map.

Proof. Assume Y is compact. Let $C \subseteq X \times Y$ be closed.
Then $(X \times Y) \setminus C$ is open.
Let $x_0 \in X \setminus \pi_1(C)$. This implies $(x_0 \times Y) \cap C = \emptyset$.
Since Y is compact and $x_0 \times Y$ is homemorphic to Y, $x_0 \times Y$ is compact.
Since $(x_0 \times Y) \cap C = \emptyset$ and $(X \times Y) \setminus C$ is open,
For each $y \in Y$, $\exists\, U_y \times V_y$ s.t. $(x_0 \times y) \in U_y \times V_y$ and $(U_y \times V_y) \cap C = \emptyset$.
This provides an open cover of $x_0 \times Y$, then there exists a finite subcover, $U_{y_1} \times V_{y_1}, ..., U_{y_n} \times V_{y_n}$, whose union is disjoint from C.
Let U $= \bigcap_{i=1}^{n} U_{y_i}$, then U is an open neighborhood of x_0 s.t. U $\subseteq X \setminus \pi_1(C)$.
Thus $\pi_1(C)$ is closed. $\square$

Now we prove the theorem.

Proof. $\Rightarrow$ Assume f is continuous.
Let $x_0 \times y_0 \in (X \times Y) \setminus G_f$. Then $f(x_0) \neq y_0$.
Since Y is Hausdorff $\exists\, V_0$ and V, disjoint open neighborhoods of y_0 and $f(x_0)$ respectively.
Then $f^{-1}(V) \times V_0$ is an open neighborhood of $x_0 \times y_0$ and since $V \cap V_0 = \emptyset$, $\left(f^{-1}(V) \times V_0\right) \subseteq (X \times Y) \setminus G_f$. Therefore G_f is closed in $X \times Y$.

$\Leftarrow$ Assume G_f is closed.
Let $x_0 \in X$. Let V be a neighborhood of $f(x_0)$ in Y.
Then $Y \setminus V$ is closed, so $X \times (Y \setminus V)$ is closed.
Since G_f is closed, $(X \times (Y \setminus V)) \cap G_f$ is closed.
Let U be the compliment of the image under π_1 of this set.
By our lemma U is open, and for each $x \in U$, $x \times f(x) \notin (X \times (Y \setminus V))$. Thus $f(U) \subseteq V$ therefore f is continuous.

$\square$